# Equilibrium Point and Prime Period 4 in Certain Piecewise Linear System of Difference Equation

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## ABSTRACT

In this article we consider two-dimensional first order piecewise linear systems of difference equations. We are specific initial conditions of solutions in some parts of $x$ axis. Equilibrium point and 4 cycles are observed in each system. We use direct calculations to show some characters of the systems and then use a single inductive statement to describe all behaviors in each system. Behaviors of each system depend on initial condition. It is either eventually equilibrium point or eventually prime period 4. We can determine that which parts of $x$ axis make solution being equilibrium point or periodic with period 4. These results will be pieces of information to understand global behaviors of family in piecewise linear system of difference equations.

*Keywords: Difference equation, Equilibrium point, Prime period solution*

## INTRODUCTION

The explanation of solution to a piecewise linear difference equation is not as simple algebraically. Especially the stability of difference equations might require Schwarzian derivative that function of the difference equations have to differentiable. But piecewise linear difference equation is not differentiable. So, the stability theorem with Schwarzian derivative cannot apply to them. An example of piecewise linear difference equation is Tent map:

$$x_{n+1} = \begin{cases} 2x_n, & x_n \le 1/2 \\ 2(1-x_n), & x_n > 1/2 \end{cases}, n = 0,1,\ldots.$$

This map can be written in the form with absolute value as $x_{n+1} = 1 - 2|x_n - 1/2|$. The map is not derivative at $1/2$. The system of difference equation with absolute value has been studied in several authors such as Lozi (1978) hypothesized a simplified version of Hénon’s map. The Lozi map is notable for having a strange attractor. Devaney (1984, 1991) investigated Gingerbreadman map. He found that the map is chaotic in certain regions and stable in others. For applications of difference equations see Cannings et al. (2005) and Cull (2006). In Grove et al. (2012) the author mentioned, an open problem, about the system of difference equations with absolute value:

$$x_{n+1} = |x_n| + ay_n + b, y_{n+1} = x_n + c|y_n| + d, n = 0,1,2,\ldots \quad (1)$$

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where the parameters $a,b,c$ and $d$ are in the set $\{-1,0,1\}$ and the initial condition $(x_0,y_0)\in R^2$. One way to explain behaviors of solutions to a piecewise linear system of difference equation is look at the pattern of solutions as the following results. Grove et. al (2012) show that every solution of a special case of system (1), $a=b=-1, c=1,$ and $d=0,$ is eventually prime period-3 solutions except for the unique equilibrium solution. Tikjha et al. (2010, 2017) found that the characters of systems, a special case of system (1), is eventually periodic with some periods or the equilibrium point. An article (Krinket & Tikjha, 2015) studied the system (1) for $a=b=-1$, $c=-1$ and $d=1$ with initial condition in *y* axis. They found that every solution is eventually prime period 4. It is worth to make the generalization of this problem by first looking at real parameter of $b$. In this article, we will investigate system of difference equation:

$$x_{n+1}=|x_n|-y_n-b, y_{n+1}=x_n-|y_n|+1, n=0,1,2,\ldots \quad (2)$$

where the parameter $b$ is 2 or 3 with specific regions in *x*-axis. In the next section we investigate system (2) with $b=2$ and initial condition belonging to a set $S_1:=\{(x,y)|-1<x<0, y=0\}$. Then we will study system (2) with $b=3$ and initial condition in $S_2:=\{(x,y)|-\infty<x<-3 \text{ and } y=0\}$. We will show that every solutions of both systems with such initial conditions are eventually prime period 4.

The following terminologies (Grove & Ladas, 2005) are used in this article:

A *system of difference equations of the first order* is a system of the form

$$x_{n+1}=f(x_n,y_n), y_{n+1}=g(x_n,y_n), n=0,1,2,\ldots \quad (3)$$

where *f* and *g* are continuous functions which map $R^2$ into *R*. A *solution* of the system of difference equations (3) is a sequence $\{(x_n,y_n)\}_{n=0}^{\infty}$ which satisfies the system for all $n\geq 0$. If we prescribe an *initial condition* $(x_0,y_0)$ in $R^2$ then $(x_1,y_1)=(f(x_0,y_0),g(x_0,y_0))$, $(x_2,y_2)=(f(x_1,y_1),g(x_1,y_1))$, … and so the solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of the system of difference equations (3) exists for all $n\geq 0$ and is uniquely determined by the initial condition $(x_0,y_0)$. A solution of the system of difference equations (3) which is constant for all $n\geq 0$ is called an *equilibrium solution*. If $(x_n,y_n)=(\bar{x},\bar{y})$ for all $n\geq 0$ is an equilibrium solution of the system (3), then $(\bar{x},\bar{y})$ is called an *equilibrium point*, or simply an *equilibrium* of the system of difference equations (3). A solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually periodic with prime period p* or *eventually prime period* *p solution* if there exists an integer $N > 0$ and $p$ is the smallest positive integer such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period $p$; that is,

$$\left(x_{n+p}, y_{n+p}\right) = (x_n, y_n) \text{ for all } n \geq N.$$

The $p$ consecutive point of the solution is called a $p$-cycle of system (3). We denote

$$\begin{pmatrix} a, & b \\ c, & d \\ e, & f \\ g, & h \end{pmatrix}$$

as 4-cycle which consists of 4 consecutive points: $(a,b)$, $(c,d)$, $(e,f)$ and $(g,h)$ in $xy$ plain. It is worth noting that solution is eventually periodic with period $p$ when *orbit* (forward iterations) contains a member of the cycle. This result would be an information to study global behaviors of system (2) for any real parameter $b$.

## MAIN RESULTS

This section we study two systems of piecewise linear difference equations. Firstly, we will investigate system (2) with $b = 2$, that is

$$x_{n+1} = |x_n| - y_n - 2, y_{n+1} = x_n - |y_n| + 1, n = 0, 1, 2, \ldots \tag{4}$$

and initial condition belong to $S_1$. It is easy to verify that an ordered pair $(-1, 0)$ is a unique equilibrium point of system (4) and system (4) has two 4-cycles:

$$P_{4.1} = \begin{pmatrix} -5, & -2 \\ 5, & -6 \\ 9, & 0 \\ 7, & 10 \end{pmatrix}, P_{4.2} = \begin{pmatrix} -1/3, & 0 \\ -5/3, & 2/3 \\ -1, & -4/3 \\ 1/3, & -4/3 \end{pmatrix}.$$

We will show that every solution of system (4) with initial condition in $S_1$ is eventually prime period 4 by the following result.

**Theorem 1.** Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be solutions of system (4) and initial condition $(x_0, y_0)$ is in $S_1$. Then the solution is eventually equilibrium point or prime period 4.

***Proof.*** We will begin with the first three iterations of the solutions as follows:

$x_1 = |x_0| - y_0 - 2 = -x_0 - 2 < 0$ and $y_1 = x_0 - |y_0| + 1 = x_0 + 1 > 0$,

$x_2 = |x_1| - y_1 - 2 = -1$ and $y_2 = x_1 - |y_1| + 1 = -2x_0 - 2 < 0$,

$x_3 = |x_2| - y_2 - 2 = 2x_0 + 1$ and $y_3 = x_2 - |y_2| + 1 = -2x_0 - 2 < 0$.

We can determine a sign of $x_1$ and $y_1$ by using the assumption of initial condition in $S_1$ that is $-1 < x_0 < 0$. We continue to iterate solution until $x_3$ and $y_3$ but we can't determine the sign of $x_3$. If we restrict $x_0$ into $-1 < x_0 \le -1/2$ then $x_4 = -1$ and $y_4 = 0$ which is equilibrium point. Thus, we can conclude that every solution of system (4) with initial condition satisfies $-1 < x_0 \le -1/2$ and $y_0 = 0$ is eventually equilibrium point. Suppose that initial condition $(x_0, y_0)$ in remain region with $x_0$ satisfies $-1/2 < x_0 < 0$ we have: $x_3 = 2x_0 + 1 > 0$. Then the next iteration $x_4 = 4x_0 + 1$ and $y_4 = 0$. Again, we can't determine $x_4$ is nonnegative or negative. If we restrict $x_0$ into $-1/4 \le x_0 < 0$, we have: $x_4 = 4x_0 + 1 \ge 0$. We continue five more iterates, we have: $x_9 = 16x_0 + 3$ and $y_9 = 16x_0 + 6 > 0$. If we restrict $x_0$ into $-3/16 \le x_0 < 0$ then $x_9 = 16x_0 + 3 \ge 0$. Thus $(x_{10}, y_{10}) = (-5, -2) \in P_{4.1}$. If $-1/4 \le x_0 < -3/16$ then $x_9 = 16x_0 + 3 < 0$. After 5 iterations we have: $(x_{14}, y_{14}) = (-5, -2) \in P_{4.1}$. Now the remain region is $-1/2 < x_0 < -1/4$ and $y_0 = 0$. We shall use a inductive statement to prove that when initial condition satisfies $-1/2 < x_0 < -1/4$ and $y_0 = 0$. The solution is eventually prime period 4. The inductive statement contains some sequences as follows:

$l_n = \dfrac{-1-2^{2n-1}}{3\times 2^{2n-1}}$, $u_n = \dfrac{1-2^{2n}}{3\times 2^{2n}}$, $a_n = \dfrac{7-2^{2n+4}}{3\times 2^{2n+4}}$ and $\delta_n = \dfrac{2^{2n}+5}{3}$. $l_n, u_n$ and $a_n$ are used to be boundaries of $x_0$. It is easy to see that limit of sequences $l_n, u_n$ and $a_n$ are $-1/3$ which will be discuss after the proof of induction. Let $P(n)$ be the following statement:

"for $x_0 \in (l_n, u_n)$,

$x_{4n+1} = -2^{2n}x_0 - \delta_n < 0$ and $y_{4n+1} = 2^{2n}x_0 + \delta_n - 1 > 0$

$x_{4n+2} = -1$ and $y_{4n+2} = -2^{2n+1}x_0 - 2\delta_n + 2 < 0$

$x_{4n+3} = 2^{2n+1}x_0 + 2\delta_n - 3$ and $y_{4n+3} = -2^{2n+1}x_0 - 2\delta_n + 2 < 0$

if $x_0 \in (l_n, l_{n+1}]$ then $x_{4n+3} = 2^{2n+1}x_0 + 2\delta_n - 3 \le 0$, and so

$x_{4n+4} = -1$ and $y_{4n+4} = 0$

if $x_0 \in (l_{n+1}, u_n)$ then $x_{4n+3} = 2^{2n+1}x_0 + 2\delta_n - 3 > 0$, and so

$x_{4n+4} = 2^{2n+2}x_0 + 4\delta_n - 7$ and $y_{4n+4} = 0$

if $x_0 \in [u_{n+1}, u_n)$ then $x_{4n+4} = 2^{2n+2}x_0 + 4\delta_n - 7 \ge 0$, and so

$x_{4n+5} = 2^{2n+2}x_0 + 4\delta_n - 9 < 0$ and $y_{4n+5} = 2^{2n+2}x_0 + 4\delta_n - 6 > 0$

$x_{4n+6} = -2^{2n+3}x_0 - 8\delta_n + 13 < 0$ and $y_{4n+6} = -2$

$x_{4n+7} = 2^{2n+3}x_0 + 8\delta_n - 13 > 0$ and $y_{4n+7} = -2^{2n+3}x_0 - 8\delta_n + 12 < 0$

$x_{4n+8} = 2^{2n+4}x_0 + 16\delta_n - 27 > 0$ and $y_{4n+8} = 0$

$x_{4n+9} = 2^{2n+4}x_0 + 16\delta_n - 29$ and $y_{4n+9} = 2^{2n+4}x_0 + 16\delta_n - 26 > 0$

if $x_0 \in [a_n, u_n)$ then $x_{4n+9} = 2^{2n+4}x_0 + 16\delta_n - 29 \ge 0$, and so

$x_{4n+10} = -5$ and $y_{4n+10} = -2$

if $x_0 \in (u_{n+1}, a_n)$ then $x_{4n+9} = 2^{2n+4}x_0 + 16\delta_n - 29 < 0$, and so

$x_{4n+10} = -2^{2n+5}x_0 - 32\delta_n + 53 < 0$ and $y_{4n+10} = -2$

$x_{4n+11} = 2^{2n+5}x_0 + 32\delta_n - 53 > 0$ and $y_{4n+11} = -2^{2n+5}x_0 - 32\delta_n + 52 < 0$

$x_{4n+12} = 2^{2n+6}x_0 + 64\delta_n - 107 > 0$ and $y_{4n+12} = 0$

$x_{4n+13} = 2^{2n+6}x_0 + 64\delta_n - 109 > 0$ and $y_{4n+13} = 2^{2n+6}x_0 + 64\delta_n - 106 > 0$

$x_{4n+14} = -5$ and $y_{4n+14} = -2$

if $x_0 \in (l_{n+1}, u_{n+1})$ then $x_{4n+4} = 2^{2n+2}x_0 + 4\delta_n - 7 < 0$". It is easy to show that $P(1)$ is true. Let $k$ be a positive integer and suppose that $P(k)$ is true. Then for

$$x_0 \in (l_{k+1}, u_{k+1}) = \left( \frac{-1-2^{2k+2}}{3\times 2^{2k+1}}, \frac{1-2^{2k+2}}{3\times 2^{2k+2}} \right),$$

we have: $x_{4k+4} = 2^{2n+2}x_0 + 4\delta_k - 7 < 0$ and $y_{4n+4} = 0$. Then

$x_{4(k+1)+1} = x_{4k+5} = |x_{4k+4}| - y_{4k+4} - 2 = -2^{2k+2}x_0 - 4\delta_k + 5 = -2^{2(k+1)}x_0 - \delta_{k+1}$,

$y_{4(k+1)+1} = y_{4k+5} = x_{4k+4} - |y_{4k+4}| + 1 = 2^{2k+2}x_0 + 4\delta_k - 6 = 2^{2(k+1)}x_0 + \delta_{k+1} - 1$.

By the boundaries of $x_0$, $(l_{n+1}, u_{n+1})$, we can determine that $x_{4(k+1)+1} < 0$ and $y_{4(k+1)+1} > 0$. Then we have: $x_{4(k+1)+2} = x_{4k+6} = |x_{4k+5}| - y_{4k+5} - 2 = -1$,

$y_{4(k+1)+2} = y_{4k+6} = x_{4k+5} - |y_{4k+5}| + 1 = -2^{2(k+1)+1}x_0 - 2\delta_{k+1} + 2$.

Since $y_{4(k+1)+2} = -2y_{4(k+1)+1}$, we have: $y_{4(k+1)+2} < 0$. Thus

$x_{4(k+1)+3} = x_{4k+7} = |x_{4k+6}| - y_{4k+6} - 2 = 2^{2(k+1)+1}x_0 + 2\delta_{k+1} - 3$,

$y_{4(k+1)+3} = y_{4k+7} = x_{4k+6} - |y_{4k+6}| + 1 = -2^{2(k+1)+1}x_0 - 2\delta_{k+1} + 2$.

Since $y_{4(k+1)+3} = y_{4(k+1)+2}$, we have: $y_{4(k+1)+3} > 0$. But the boundaries of $x_0$, cannot determine $x_{4(k+1)+3}$ whether it is non-negative or negative. We follow the inductive statement by considering

$$x_0 \in (l_{k+1}, l_{k+2}] = \left( \frac{-1-2^{2k+1}}{3\times 2^{2k+1}}, \frac{-1-2^{2k+3}}{3\times 2^{2k+3}} \right]$$

which make $x_{4(k+1)+3} \le 0$. So $x_{4(k+1)+4} = -1$ and $y_{4(k+1)+4} = 0$. That is, the solution is an equilibrium point. For the remain case

$$x_0 \in (l_{k+2}, u_{k+1}) = \left( \frac{-1-2^{2k+3}}{3\times 2^{2k+3}}, \frac{1-2^{2k+2}}{3\times 2^{2k+2}} \right)$$

which make $x_{4(k+1)+3} > 0$. We have: $x_{4(k+1)+4} = x_{4k+8} = 2^{2(k+1)+2}x_0 + 4\delta_{k+1} - 7$ and $y_{4(k+1)+4} = y_{4k+8} = 0$. We continue this process following inductive statement to verify *P*(*k*+1) is true by direct computation. We use the induction to conclude that *P*(*n*) is true for every positive integer *n*. By the inductive statement *P*(*n*), we can conclude as follows:

1. If $x_0 \in (l_n, l_{n+1}]$ for some *n*, then the solution is equilibrium point in the next iteration.
2. If $x_0 \in [u_{n+1}, u_n)$ for some *n*, then the solution is eventually prime period 4 ($P_{4.1}$).

We note that sequence $l_n$ is in the left side of its limit and sequences $u_n, a_n$ are in the right side of their limit. As mention earlier that the limit of $l_n, u_n$ and $a_n$ are $-1/3$. So the solution of the system is eventually equilibrium point when choosing $x_0$ in left hand side of $-1/3$ and is eventually prime period 4 when choosing $x_0$ in right hand side of $-1/3$. For $x_0 = -1/3$, we have: $(x_0, y_0) = (-1/3, 0) \in P_{4.2}$. □

Next, we will study the system (2) with $b = 3$, that is

$$x_{n+1} = |x_n| - y_n - 3, y_{n+1} = x_n - |y_n| + 1, n = 0, 1, 2, \dots \quad (5)$$

and initial condition belong to $S_2$. It is easy to verify that an ordered pair $(-1, -1)$ is a unique equilibrium point of system (5) and system (5) has two 4-cycles:

$$P_{4.1}^* = \begin{pmatrix} -5 & ,-1 \\ 3 & ,-5 \\ 5 & ,-1 \\ 3 & ,\ 5 \end{pmatrix}, P_{4.2}^* = \begin{pmatrix} 1 & ,-3 \\ 1 & ,-1 \\ -1 & ,\ 1 \\ -3 & ,-1 \end{pmatrix}.$$

Let $(x_0, y_0) \in S_2$. So $x_0 \in (-\infty, -3)$ and $y_0 = 0$. Then

$x_1 = |x_0| - y_0 - 3 = -x_0 - 3 > 0$ and $y_1 = x_0 - |y_0| + 1 = x_0 + 1 < 0$,

$x_2 = |x_1| - y_1 - 3 = -2x_0 - 7$ and $y_2 = x_1 - |y_1| + 1 = -1$.

If $x_0 \in \left[ -\frac{7}{2}, -3 \right)$ then $x_2 \le 0$ and so

$x_3 = |x_2| - y_2 - 3 = 2x_0 + 5 < 0$ and $y_3 = x_2 - |y_2| + 1 = -2x_0 - 7 \le 0$, $x_4 = |x_3| - y_3 - 3 = -1$ and $y_4 = x_3 - |y_3| + 1 = -1$. After 4 iterations, the solution will be equilibrium point. Suppose that $x_0 \in \left(-\infty, -\frac{7}{2}\right)$ then $x_2 > 0$, we have: $x_3 = |x_2| - y_2 - 3 = -2x_0 - 9$ and $y_3 = x_2 - |y_2| + 1 = -2x_0 - 7 > 0$.

If $x_0 \in \left(-\infty, -\frac{9}{2}\right]$ then $x_3 \ge 0$. Thus $(x_4, y_4) = (-5, -1) \in P_{4.1}^*$.

If $x_0 \in \left(-\frac{9}{2}, -\frac{7}{2}\right)$ then $x_3 < 0$. We have: $x_4 = 4x_0 + 13 < 0$ and $y_4 = -1$. Then $x_5 = |x_4| - y_4 - 3 = -4x_0 - 15$ and $y_5 = x_4 - |y_4| + 1 = 4x_0 + 13 < 0.$

If $x_0 \in \left[-\frac{15}{4}, -\frac{7}{2}\right)$ then $x_5 \le 0$. Thus $(x_6, y_6) = (-1, -1)$. In this case the solution will be equilibrium point in 6 iterations. We can summarize all above results that the solution of system (5) is eventually prime period 4 ($P_{4.1}^*$) when $x_0 \in \left(-\infty, -\frac{9}{2}\right]$ and is eventually equilibrium point when $x_0 \in \left[-\frac{15}{4}, -3\right)$. The remain interval that we will investigate, is $\left(-\frac{9}{2}, -\frac{15}{4}\right)$. For $x_0 \in \left(-\frac{9}{2}, -\frac{15}{4}\right)$, we have $x_5 = -4x_0 - 15 > 0$. Then we will use an inductive statement to prove that the solution is eventually equilibrium point or prime period 4 by using the following statement:

"for $x_0 \in (l_n, u_n)$,

$x_{4n+2} = -2^{2n+1}x_0 - \delta_n$ and $y_{4n+2} = -1$

if $x_0 \in [c_n, u_n)$ then $x_{4n+2} \le 0$ and so

$x_{4n+3} = 2^{2n+1}x_0 + \delta_n - 2 < 0$ and $y_{4n+3} = -2^{2n+1}x_0 - \delta_n < 0,$

$x_{4n+4} = -1$ and $y_{4n+4} = -1$

if $x_0 \in (l_n, c_n)$ then $x_{4n+2} > 0$ and so

$x_{4n+3} = -2^{2n+1}x_0 - \delta_n - 2$ and $y_{4n+3} = -2^{2n+1}x_0 - \delta_n > 0$

if $x_0 \in (l_n, l_{n+1}]$ then $x_{4n+3} \ge 0$ and so

$x_{4n+4} = -5$ and $y_{4n+4} = -1$

if $x_0 \in (l_{n+1}, c_n)$ then $x_{4n+3} < 0$ and so

$x_{4n+4} = 2^{2n+2}x_0 + 2\delta_n - 1 < 0$ and $y_{4n+4} = -1$

$x_{4n+5} = -2^{2n+2}x_0 - 2\delta_n - 1$ and $y_{4n+5} = 2^{2n+2}x_0 + 2\delta_n - 1 < 0$

if $x_0 \in [u_{n+1}, c_n)$ then $x_{4n+5} \le 0$ and so

$x_{4n+6} = -1$ and $y_{4n+6} = -1$

if $x_0 \in (l_{n+1}, u_{n+1})$ then $x_{4n+5} > 0$"

where $l_n = \dfrac{-2^{2n+1} - 1}{2^{2n-1}}, u_n = \dfrac{-2^{2n+2} + 1}{2^{2n}}, c_n = \dfrac{-2^{2n+3} + 1}{2^{2n+1}}$ and $\delta_n = 2^{2n+3} - 1$. We note that the limit of sequences $l_n, u_n$ and $a_n$ are $-4$. For $x_0 = -4$ we have: $(x_1, y_1) = (1, -3) \in P_{4.2}$ . The solution will be equilibrium point when selecting $x_0 \in (-4, -3)$ and prime period 4 when selecting $x_0 \le -4$. Hence, we have the following theorem.

**Theorem 2.** Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be solutions of system (5) and initial condition $(x_0, y_0)$ is in $S_2$. Then the solution is eventually equilibrium point or prime period 4.

## CONCLUSIONS

The solutions of system (4) with initial condition in $S_1$ and system (5) with initial condition in $S_2$ have the same behaviors that is both systems have two attractors, equilibrium point and 4-cycles. But in article (Krinket & Tikjha, 2015), 4 cycles are only detected attractor to the system. This happened because of the restriction of initial conditions and we chose initial condition in a basin of attraction of 4 cycles. So the attractor of the system that studied in article (Krinket & Tikjha, 2015), may be more than one attractor. This is a basic information about the global behaviors (by looking at initial condition in *x* axis) of this family of piecewise linear system. In future work, we will extend the initial condition into quadrants or entire *xy*-plan.

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