Common Fixed Point of Self Mappings under A-Contraction in b-Metric Spaces

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ABSTRACT

In this paper, we study the existence of common fixed point for a sequence of self mappings under A-contraction type concept in b-metric spaces. Our result extend and generalize the result derived by Akram et al. and many others.

Keywords: fixed point theory, metric spaces, b-metric spaces, A-contraction

INTRODUCTION

Metric fixed point theory the first important and significant result was proved by Banach in 1922 for contraction mapping in complete metric space. Many authors studied this principle and its generalizations in different types of contractions in metric spaces. Akram et al. (2008) introduced a new class of contraction maps, call A-contractions, which includes the contractions studied by Bianchini (1972), Khan (1978/79), Reich (1971), and Kannan (1968). They also have shown that a metric space is complete if and only if it has a fixed point property for A-contractions. There have appeared many generalizations of metric spaces. One is the concept of a b-metric space which introduced by Czerwik (1993) and Bakhtin (1989). Recently, Pankaj et al. (2014), gave some results related fixed point theorem in b-metric spaces. They have shown the extension theorem given by Reich (1971) and Hardy and Rogers (1973) to the b-metric spaces. Otherwise, Mehmet and Hukmi (2013) and Hussain et al. (2012), gave some results for contraction in b-metric space. Motivate Pankaj et al. (2014), Mehmet and Hukmi (2013), Akram et al. (2008) and reference therein, we give a fixed point theorem for a sequence self mappings with respect to A-contraction in a b-metric space in our main theorem.

PRELIMINARIES

Definition 2.1 (Aydi et al. (2012))

Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to R^+$ (nonnegative real numbers) is called a b-metric provided that for all $x, y, z \in X$, the following conditions are satisfied:

1) $d(x, y) = 0$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, z) = s[d(x, y) + d(y, x)]$.

A pair (X,d) is called b-metric space.

From the above definition the class of b-metric spaces is larger than the class of (usual) metric spaces. For $s=1$ it reduces into (usual) metric space. However, Czewik (1993) and Bakhtin (1989) have shown that a b-metric on X need not to be a metric on *X*. We can give some examples of b-metric space as the following.

Example 2.2 (Berinde (1993))

The space $L_p(0<p<1)$ of all real functions $x(t), t\in[0,1]$ such that $\int_0^1 |x(t)|^p < \infty$, is b-metric space if we take $d(x,y)=\left(\int_0^1 |x(t)-y(t)|^p\, dt\right)^{\frac{1}{p}}$, for each $x,y\in L_p$.

Example 2.3 (Aydi et al. (2012))

Let $X=\{0,1,2\}$ and $d(2,0)=d(0,2)=m\geq 2$, $d(0,1)=d(1,2)=d(2,1)=1$ and $d(0,0)=d(1,1)=d(2,2)=0$. Then $d(x,y)\leq \frac{m}{2}[d(x,2)+d(2,y)]$ for all $x,y,z\in X$. If $m>2$, the ordinary triangle inequality does not hold.

Definition 2.4 (Pankaj et al.(2014))

Let (X,d) is a b-metric space. Then a sequence $\{x_n\}$ in *X* is called Cauchy sequence if and only if for every $\varepsilon>0$, there exists $n(\varepsilon)\in N$, such that for all m, n $\geq n(\varepsilon)$ we have $d(x_n, x_m)<\varepsilon$.

Definition 2.5 (Mehmet and Hukmi (2013))

Let (X,d) is a b-metric space. Then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x\in X$, such that for every $\varepsilon>0$, there exists $n(\varepsilon)\in N$, such that $d(x_n, x)<\varepsilon$, for all $n\geq n(\varepsilon)$. In this case we write $\lim_{n\to\infty} x_n = x$.

Definition 2.6 (Pankaj et al.(2014))

The b-metric space (X,d) is complete if every Cauchy sequence in X is convergent sequence in X.

Definition 2.7 (Mehmet and Hukmi (2013))

Let X be any nonempty set and T: $X\to X$ a selfmap. We say that $x\in X$ is a fixed point of T if $T(x)=x$, for the convenience we write $Tx=x$. Recall $T^n x$ the n^{th} iterative of x under T. For any $x_0\in X$, the sequence $\{x_n\}_{n\geq 0}\subset X$ is given by

$$x_n = Tx_{n-1} = T^n x_0,\ n = 1,2,3,\ldots \tag{1}$$

is called the sequence of successive approximations with the initial value x_0. It is known as the Picard iteration starting at x_0.

Definition 2.8 (Akram et al. (2008))

A self-map T on a metric space X is said to be

1) K-contraction if there exists a number $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq s[d(Tx, x)+d(Ty, y)] \quad \text{for all } x, y \in X.$$

2) B-contraction if there exists a number $b \in [0,1)$ such that

$$d(Tx, Ty) \leq b \max\{d(Tx, x), d(Ty, y)\} \text{ for all } x, y \in X.$$

3) R-contraction if there exists non-negative numbers a, b, c satisfying $a+b+c \leq 1$ such that

$$d(Tx, Ty) \leq a\, d(Tx,x) + b\, d(Ty,y) + c\, d(x, y) \text{ for all } x, y \in X.$$

On the other hand, Akram et al.[1] introduced a new class of contraction maps, call A-contraction, which is a proper superclass of Kannan's [9], Bianchini's [5] and Reich's [13] contractions type as the following definition.

Definition 2.9 (Akram et al. (2008))

A self-map T on a metric space X is said to be A-contraction if it satisfies the condition:

$$d(Tx, Ty) \leq \alpha\,(d(x, y), d(x,Tx), d(y,Ty))$$

for all $x, y \in X$ and some $\alpha \in A$ which A be the set of all functions $\alpha : R_+^3 \to R_+$ satisfying

1) α is continuous on the set R_+^3 with respect to the Euclidean metric on R^3.

2) $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all $a, b \in R^+$.

MAIN RESULT

Theorem 1. Let $\alpha \in A$ and $\{T_n\}$ be a sequence of self mappings on a complete b-metric space (X, d) such that $d(T_i x, T_j y) \leq \alpha\left(d(x, y), d(T_i x, x), d(T_j y, y)\right)$ holds for each $x, y \in X$. Then $\{T_n\}$ has a unique common fixed point in X.

Proof. Let $z_0 \in X$ and each $n \in N$, we define $z_n = T_n z_{n-1}$. Since $\alpha \in A$, we have

$$\begin{aligned} d(z_1, z_2) &= d(T_1 z_0, T_2 z_1) \\ &\leq \alpha\,(d(z_0, z_1), d(T_1 z_0, z_0), d(T_2 z_1, z_1)) \end{aligned} \tag{2}$$

$$= \alpha\ (d(z_0, z_1), d(z_1, z_0), d(z_2, z_1))$$
$$\leq k_1 d(z_0, z_1),$$

for some $k_1 \in [0,1)$. Similarly,

$$\begin{aligned} d(z_2, z_3) &= d(T_2 z_1, T_3 z_2) \\ &\leq \alpha\ (d(z_1, z_2), d(z_1, T_2 z_1), d(z_2, T_3 z_2)) \\ &= \alpha\ (d(z_1, z_2), d(z_1, z_2), d(z_2, z_3)) \\ &\leq k_2 d(z_1, z_2), \end{aligned} \tag{3}$$

for some $k_2 \in [0,1)$.

Using (2) and (3) we have $d(z_2, z_3) \leq k_1 k_2 d(z_0, z_1)$ for some $k = k_1 k_2 \in [0,1)$. In general, we obtain $d(z_n, z_{n+1}) \leq k^n d(z_0, z_1)$ for some $k \in [0,1)$. Therefore, $\{z_n\}$ is a Cauchy sequence in a complete metric space X. Then there exists some $z \in X$ such that $\lim_{n\to\infty} z_n = z$. Next, we show that z is a fixed point of T_n for all $n \in N$. For each $n \in N$ be fixed and

$$\begin{aligned} d(z, T_n z) &\leq s[d(z, z_{m+1}) + d(z_{m+1}, T_n z)] \\ &\leq sd(z, z_{m+1}) + s^2[d(z_{m+1}, T_{m+1} z_m) + d(T_{m+1} z_m, T_n z)] \\ &= sd(z, z_{m+1}) + s^2 d(z_{m+1}, T_{m+1} z_m) + s^2 d(T_{m+1} z_m, T_n z) \\ &\leq sd(z, z_{m+1}) + s^2 \alpha(d(z_m, z), d(z_m, T_{m+1} z_m), d(z, T_n z)) \\ &= sd(z, z_{m+1}) + s^2 \alpha(d(z_m, z), d(z_m, z_{m+1}), d(z, T_n z)). \end{aligned}$$

Taking limit as $m \to \infty$, we obtain

$$d(z, T_n z) \leq sd(z, z) + s\alpha(d(z, z), d(z, z), d(z, T_n z)) \leq 0.$$

Hence, $T_n z = z$, for all $n \in N$. We have z is a common fixed point of T_n for all $n \in N$. Let z and w are two common fixed point of T_n. Then

$$\begin{aligned} d(z,w) &= d(T_n z, T_n w) \\ &\leq \alpha(d(z, w), d(z, T_n z), d(w, Tw)) \\ &\leq \alpha(d(z, w), 0, 0) \\ &\leq k(0) \\ &\leq 0. \end{aligned}$$

Therefore, T_n have a unique common fixed point. This completes the proof.

CONCLUSIONS

Our result extends and generalizes theorem in Aram et al. (2008). Furthermore, the result in Aram et al. (2008), indicate A-contraction is a proper superclass of Kannan's (1968) and Reich's (1971) contractions type, which Mehmet and Hukmi (2013) and Pankaj et al.(2014) are also generalized by our theorem in the case b-metric spaces.

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REFERENCES

Akram, M., Zafar, A.A., and Siddiqui, A.A. (2008). A general class of contractions: A-contractions. *Novi Sad Journal of Mathematics,* 30, 25-33.

Aydi, H., Bota, M. Karapinar, E. and Mitrovic, S. (2012). A fixed point theorem for set-valued quasi-contractions on b-metric spaces, *Fixed Point Theory Appl*, 88.

Bakhtin, I.A. (1989). The contraction mapping principle in quasi metric space, *Funct. Anal. Unianowsk Gos. Ped. Inst*, 30, 26-37.

Berinde, V. (1993). Generalized contractions in quasimetric spaces, *Seminar on Fixed Point Theory, Babes-Bolyai Univ. Cluj-Napoca*, 3, 3-9.

Bianchini, R. (1972). Su un problems dis Reich riguardante la teoria dei punti fissi. *Boll Un. Mat. Ital,* 5, 103-108.

Czerwik, s. (1993). Contraction mappings in b-metric space. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1, 5-11.

Hardy, G. and Rogers, T. (1973). A generalization of fixed point theorem of Reich. *Canadian Mathematical Bulletin*, 16, 201-206.

Hussain, N., Doric, D., Kadelburg, Z., and Radenovic, S. (2012). Suzuki-type fixed point results in metric type space, *Fixed Point Theory Appl., doi:10.1186/1687-1812-2012-126.*

Kanan, R. (1968). Some results on fixed point. *Bull. Caleutta Math. Soc*, 60, 71-76.

Khan, M. (1978/79). *On fixed point theorems*. *Math. Japonica*, 23, 201-204.

Mehmet, K. and Hukmi, K. (2013). On some well known fixed point theorem in b-Metric spaces. *Turkish Journal of Analysis and Number theory*, 1, 13-16.

Pankaj, K., Shweta, S. and Banerjee, S. (2014). Some Fixed Point in b-metric Space. *Turkish Journal of Analysis and Number Theory*, 2, 19-22.

Reich, S. (1971). *Kannan's Fixed Point Theorem*. *Boll. Un. Math. Ital*, 4, 1-11.