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Semigroups with Some Conditions which do not Admit a Distributive Near – ring Structure

Manoj Siripitukdet

Department of Mathematics, Faculty of Science,
Naresuan University, Phitsanulok 65000 Thailand
E- mail: manojs@nu.ac.th

ABSTRACT

In this paper, we considered semigroups with some conditions and showed that they did not admit a distributive near – ring structure.

Keywords: semigroups, distributive near – ring

INTRODUCTION

A system $(S,+,\cdot)$ is called a *(right) near – ring* (Pilz, 1983) if (i) $(S,+)$ is a group, (ii) $(S,\cdot)$ is a semigroup and (iii) $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x, y, z \in S$ (a right distributive law). For a (right) near – ring $(S,+,\cdot)$, an element $d \in S$ is called *distributive* (Pilz, 1983) if $d\cdot(x+y) = d\cdot x + d\cdot y$ for all $x, y \in S$. Let $D = \{d \in S \mid d$ is distributive$\}$. A (right) near – ring S is called *distributive* (Pilz, 1983) if $S = D$. Then, clearly, S is a distributive near – ring if and only if for all $d \in S$, d is distributive.

An element a of a semigroup S is a *zero* if $ax = xa = a$ for all $x \in S$ and we denote a by 0.

For any semigroup S, let $S^0 = S$ if S has a zero and S contains more than one element, and otherwise, let S^0 be the semigroup with zero 0 adjoined. For a symbol S^1, we define $S^1 = S$ if S has an identity, otherwise, let $S^1 = S \cup \{1\}$ if S has no identity. A semigroup S is said to *admit a ring structure* (Satyanarayana, 1981) if there exists some ring R such that S^0 is isomorphic to the semigroup $(R,\cdot)$ where $\cdot$ is the multiplication of R, or equivalently, there exists an operation $+$ on S^0 such that $(S^0,+,\cdot)$ is ring where $\cdot$ is the operation on S^0.

A semigroup admitting a distributive near – ring structure is defined similarly.

Let $SDN := \{S \mid S$ is a semigroup admitting a distributive near – ring structure$\}$,

$SR := \{S \mid S$ is a semigroup admitting a ring structure$\}$.

Clearly, $SR \subseteq SDN$. That is, if S is a semigroup admitting a ring structure, then S admits a distributive near – ring structure. Semigroups admitting a ring structure have long been studied. For example, see (Chu and Shyr, 1980; Keprasit and Siripitukdet, 2002; Lawson, 1969; Peinado, 1970; Satyanarayana, 1981).

For a semigroup S, let $E(S)$ denote the set of all idempotents of S. Then $(E(S), \leq)$ is a partially ordered set (Howie, 1976) where $\leq$ is defined by for $e, f \in E(S), e \leq f$ if and only if $e = ef = fe$.

In this paper, semigroups with some conditions are considered and investigated when or whether they admit a distributive near – ring structure.

The next proposition is used in this paper.

Proposition 1 (Siripitukdet, 2001). *Let* (S, +, ·) *be a ditributive near – ring. Then the following statements hold:*

(i) $0x = x0 = 0$ *for all* $x \in S$ *where* 0 *is the identity of the group S.*

(ii) $-(-x) = x$ *for all* $x \in S$.

(iii) $x(-y) = (-x)y = -(xy)$ *and* $(-x)(-y) = xy$ *for all* $x, y \in S$.

(iv) *For all* $x, y, u, v \in S$, $xy + uv = uv + xy$.

(v) *If* $S = S^2$ *where* $S^2 = \{xy \mid x, y \in S\}$, *then S is a ring.*

(vi) *If S has a left or right multiplicative identity, then S is a ring (hence S has a multiplicative identity, then S is a ring).*

(vii) *For each* $x \in S$, $(xS, +)$ *and* $(Sx, +)$ *is a group where* $xS = \{xs \mid s \in S\}$ *and* $Sx = \{sx \mid s \in S\}$.

Theorem 2. *Let S be a semigroup with zero* 0 *and* $E(S) > 2$.

(i) *If the product of every two distinct elements in* $E(S)$ *is* 0, *then S does not admit a distributive near – ring structure.*

(ii) *If the elements in* $E(S)$ *form a chain, then S does not admit a distributive near – ring structure.*

Proof. (i) Assume that S admits a distributive near – ring structure. Then there is a binary operation $+$ on S such that $(S, +, \cdot)$ is a distributive near – ring where $\cdot$ is the given binary operation of S. Let $e, f \in E(S) \setminus \{0\}$ be such that $e \neq f$. Then $(e+f)^2 = (e+f)(e+f) = e^2 + ef + fe + f^2 = e + f$. Thus $e + f \in E(S)$ and $e + f \neq f$. By assumption, $0 = (e+f)f = ef + f^2 = f$, a contradiction. Hence S does not admit a distributive near – ring structure.

(ii) Recall that $(E(S), \leq)$ is a partially ordered set where $\leq$ is defined by for $e, f \in E(S)$, $e \leq f$ if and only if $e = ef = fe$.

Assume that S admits a distributive near – ring structure. Then there is a binary operation + on S such that $(S, +, \cdot)$ is a distributive near – ring where $\cdot$ is the given binary operation of S. Let $e, f \in E(S)\setminus\{0\}$ be two distinct elements such that $e < f$. Then $e = ef = fe$. Now

$$(f-e)^2 = (f-e)(f-e) = f^2 - ef - fe + e^2 = f - e - e + e = f - e,$$

so $f - e \in E(S)$. By assumption, $e < f - e$ or $f - e < e$ or $e = f - e$.

If $e < f - e$, then $e = e(f-e) = ef - e^2 = e - e = 0$, a contradiction.

If $f - e < e$, then $f - e = (f-e)e = fe - e = e - e = 0$, a contradiction.

Hence $e = f - e$. Then $e = e^2 = e(f-e) = ef - e = e - e = 0$, a contradiction. Therefore S does not admit a distributive near – ring. □

MAIN RESULTS

Some conditions are given for a semigroup with zero and show that $S \in SDN$ if and only if $|S| \le 2$.

Theorem 3. *Let S be a semigroup with zero* 0. *Assume that*

(i) *for* $x, y \in S$, $xS^1 \subseteq yS^1$ *or* $yS^1 \subseteq xS^1$ *and*

(ii) *for* $x, y \in S$, $xS^1 = yS^1$ *implies* $x = y$.

Then S admits a distributive near – ring structure if and only if $|S| \le 2$.

Proof. Assume that S admits a distributive near – ring structure. Then there is a binary operation $+$ on S such that $(S, +, \cdot)$ is a distributive near – ring where $\cdot$ is the given binary operation of S. Suppose that $|S| > 2$. Let x, y be two nonzero distinct elements in S. By assumption (i), $xS^1 \subseteq yS^1$ or $yS^1 \subseteq xS^1$. Without loss of generality, we may assume that $xS^1 \subseteq yS^1$. Since $x \in xS^1$, $x = ys_1$ for some $s_1 \in S^1$. Since $x \ne y$, $s_1 \ne 1$.

Thus $$x = ys_1 \in yS \tag{1}$$

By the assumption (i), $(x+y)S^1 \subseteq xS^1$ or $xS^1 \subseteq (x+y)S^1$.

Case 1. $(x+y)S^1 \subseteq xS^1$. Since $x + y \in (x+y)S^1$, $x + y = xs_2$ for some $s_2 \in S^1$. Thus $x + y \in xS$ and $y = xs_2 - x$. From (1), $x = ys_1 = (xs_2 - x)s_1 = xs_2s_1 - xs_1$ $= x(s_2s_1 - s_1) \in xS$. Since $(xS, +)$ is a group, $y = (-x) + (x+y) \in xS$. Thus $yS^1 \subseteq xS^1$. By the condition (ii), we have $x = y$, a contradiction.

Case 2. $xS^1 \subseteq (x+y)S^1$. Since $x \in xS^1 \subseteq (x+y)S^1$, $x = (x+y)s_3$ for some $s_3 \in S$.

Thus $$x \in (x+y)S. \tag{2}$$

By the condition(i), $(x+y)S^1 \subseteq yS^1$ or $yS^1 \subseteq (x+y)S^1$.

Subcase 2.1. $(x+y)S^1 \subseteq yS^1$. Since $x+y \in (x+y)S^1 \subseteq yS^1$, we have
$$x+y = ys_4 \text{ for some } s_4 \in S. \tag{3}$$

Let $s_5 = -s_1 + s_4$. Then $s_5 \in S$. Now $ys_4s_5 = (x+y)s_5 \in (x+y)S$. By (2), we have that $xs_5 \in (x+y)S$. Since $((x+y)S, +)$ is a group and $ys_4s_5,\ xs_5 \in (x+y)S$, we have $-(xs_5)+(ys_4s_5) \in (x+y)S$. By (3) and (1), we obtain that
$$y = -x + ys_4 = -(ys_1) + ys_4 = y(-s_1 + s_4) = ys_5 = (-x + ys_4)s_5$$
$$= -(xs_5) + (ys_4s_5) \in (x+y)S.$$

Thus $yS^1 \subseteq ((x+y)S)S^1 \subseteq (x+y)S^1$. By the condition (ii), $y = x+y$ so $x = 0$, a contradiction.

Subcase 2.2. $yS^1 \subseteq (x+y)S^1$.

Since $y \in yS^1 \subseteq (x+y)S^1$, $y = (x+y)s_6$ for some $s_6 \in S$.

Thus $$y \in (x+y)S. \tag{4}$$

From (2) and (4) and $((x+y)S, +)$ is a group, we get that $x+y \in (x+y)S$. Thus $x+y = (x+y)s_7$ for some $s_7 \in S$. From (1), we have that
$$x+y = xs_7 + ys_7 = ys_1s_7 + ys_7 = y(s_1s_7 + s_7) \in yS.$$

Therefore $(x+y)S^1 \subseteq (yS)S^1 \subseteq yS^1$. By the condition (2), $x+y = y$ so $x = 0$, a contradiction. Therefore $|S| \le 2$.

Conversely, assume that $|S| \le 2$. If $|S| = 1$, then $S = \{0\}$ so we are done. Assume that $|S| = 2$. Let $S = \{0, x\}$. Then $x^2 = x$ or $x^2 = 0$, so $(S, \otimes) \cong (\mathbb{Z}_2, \cdot)$ where $\otimes$ is the binary operation of S and $\cdot$ is the usual multiplication of $\mathbb{Z}_2$ or S is a zero semigroup. Hence S admits a ring structure. □

Corollary 4. *Semigroups $[0,1)$ and $[0,1]$ under the usual multiplication do not admit a distributive near – ring structure.*

Proof. Let $S \in \{[0,1), [0,1]\}$. Then $S^1 = [0,1]$. If for $x, y \in S$, then $x \le y$ or $y \le x$ and so $xS^1 \subseteq yS^1$ or $yS^1 \subseteq xS^1$. If for $x, y \in S$ and $xS^1 \subseteq yS^1$, then $[0, x] = [0, y]$ so $x = y$. By Theorem 3, S does not admit a distributive near – ring structure. □

Remark: Under the usual multiplication, we see that $[0,1)\cong(1,\infty)^0$ and $[0,1]\cong[1,\infty)^0$ by defining $f(x)=\frac{1}{x}$ for all $x\in(0,1)$ and $f(0)=0$ and $g(x)=\frac{1}{x}$ for all $x\in(0,1]$ and $g(0)=0$, respectively. By Corollary 4, $(1,\infty)$ and $[1,\infty)$ do not admit a distributive near – ring structure.

Theorem 5. *Let S be a semigroup without zero. Assume that*

(i) *for* $x, y\in S$, $xS^1\subseteq yS^1$ *or* $yS^1\subseteq xS^1$ *and*

(ii) *for* $x, y\in S$, $xS^1=yS^1$ *implies* $x=y$.

Then S adimits a distributive near – ring structure if and only if $|S|=1$.

Proof. Assume that S admits a distributive near – ring structure. Then there is an operation $+$ on S^0 such that $(S^0,+,\cdot)$ is a distributive near – ring where $\cdot$ is the given operation on S^0. Suppose that $|S|>1$. Let x, y be distinct elements in S. By (i), $xS^1\subseteq yS^1$ or $yS^1\subseteq xS^1$. We may assume that $xS^1\subseteq yS^1$. By the same proof in Theorem 3, we have that $x=ys_1\in yS$ and $(x+y)S^1\subseteq xS^1$ or $xS^1\subseteq(x+y)S^1$.

Case 1. $(x+y)S^1\subseteq xS^1$. Using the same proof as in Theorem 3 case 1, we have that $(x+y)\in xS\subseteq xS^0$ and $x\in xS\subseteq xS^0$. Since $(xS^0,+)$ is a group, $y=-x+(x+y)\in xS^0$ which implies that $y\in xS$. Thus $yS^1\subseteq xSS^1\subseteq xS^1$. By (ii), $x=y$, a contradiction.

Case 2. $xS^1\subseteq(x+y)S^1$. Using the same proof as in Theorem 3, we have $x\in(x+y)S$. By (i) $(x+y)S^1\subseteq yS^1$ or $yS^1\subseteq(x+y)S^1$.

Subcase 2.1. $(x+y)S^1\subseteq yS^1$. Modify the proof of Theorem 3 in subcase 2.1 by using the fact that $(x+y)S\subseteq(x+y)S^0$ and $((x+y)S^0,+)$ is a group, we have that $yS^1\subseteq(x+y)S^1$. By (ii), $y=x+y$ so $x=0$, a contradiction.

Subcase 2.2. $yS^1\subseteq(x+y)S^1$. Modify the proof of Theorem 3 in subcase 2.2 by using the fact that $((x+y)S^0,+)$ is a group, we have that $(x+y)S^1\subseteq yS^1$. By (ii), $x+y=y$ so $x=0$, a contradiction.

Therefore $|S|=1$.

The converse is obvious. □

Corollary 6. *Semigroups* (0,1) *and* (0,1] *under the usual multiplication do not admit a distributive near – ring structure.*

Proof. It is similar to the proof of Corollary 4. □

The last section, we give an example satisfying the condition (i) in Theorem 2.

Example. Let R be a ring with identity $1 \neq 0$ where 0 is the identity of the group R and let n be a positive integer greater than 1. For $i, j \in \{1, 2, \ldots, n\}$, let $E^{ij} = \left[\left(e^{ij}\right)_{st}\right]$ be a matrix in $M_n(R)$ defined by

$$\left(e^{ij}\right)_{st} = \begin{cases} 1 & \text{if } s = i \text{ and } t = j, \\ 0 & \text{otherwise} \end{cases}$$

i.e.

$$E^{ij} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \leftarrow \text{row } i$$

$$\uparrow$$

$$\text{column } j$$

For $x \in R$,

$$xE^{ij} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & x & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \leftarrow \text{row } i$$

$$\uparrow$$

$$\text{column } j$$

Let $S = \left\{ xE^{ij} \,\middle|\, x \in R \text{ and } i, j \in \{1, 2, 3, \ldots, n\} \right\}$. Then, clearly, $(S, \cdot)$ is a semigroup under the usual multiplication of matrices.

For $i, j, s, t \in \{1, 2, \ldots, n\}$ and $x, y \in R$,

$$\left(xE^{ij}\right)\left(yE^{st}\right) = \begin{cases} xyE^{it} & \text{if } j = s, \\ 0 & \text{if } j \neq s. \end{cases}$$

It is clear that the set of all nonzero-idempotents of S is $\left\{E^{ii} \mid i \in \{1, 2, \ldots, n\}\right\}$. Clearly, for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, $\left(E^{ii}\right)\left(E^{jj}\right) = \bar{0}$ where $\bar{0}$ is the zero matrix in $M_n(R)$. By Theorem 2 (i), S does not admit a distributive near – ring structure.

CONCLUSION

Some conditions (in Theorem 3) for a semigroup with zero and without zero are investigated and showed that a semigroups with zero (without zero) satisfying this conditions belong to the class SDN if and only if $|S| \leq 2$ ($|S| = 1$).

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