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The perfect matching or the near-perfect matching of the union of a Kneser graph and a Johnson graph

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ABSTRACT

The union of a Kneser graph and a Johnson graph, denoted by $L(n, k)$, has the k-element subsets of an n-element set as vertices for $n \in \mathbb{N}$ and $k \in \mathbb{N}$, with two vertices are adjacent if the sets are disjoint or their intersection has size k -1. We show that $L(n, k)$ has a perfect matching or a near-perfect matching for all n and k. Particularly, we find its matching number and edge cover number.

Keywords: Kneser graph, Johnson graph, Perfect-Matching, Near-perfect-matching, Edge cover number

INTRODUCTION

The Kneser graph $K(n, k)$ is the graph whose vertices correspond to the k-element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. In (Etzion and Bitan, 1996) defined the Johnson graph $J(n, k)$ to be the graph whose the vertex set $V(J(n, k))$ equals $V(K(n, k))$, and two k-element subsets are adjacent if and only if their intersection has size k -1 . Moreover, they studied on the chromatic number, coloring and code of the Johnson graph. In (Kim and Nakprasit, 2004) studied chromatic numbers and independent numbers of $(K^2(2k + 1, k))$. Next, (Yonsomeng, 2009) defined the union of a Kneser graph and a Johnson graph $L(n, k)$ has $V(L(n, k))$ equals $V(K(n, k))$. In $L(n, k)$, the distinct vertices u and v are adjacent in $L(n, k)$ if and only if $u \cap v = \emptyset$ or $|u \cap v| = k - 1$. In fact, $K^2(2k+1, k) \cong L(2k+1, k)$. Moreover, she studied clique numbers of $L(n, k)$ for $k \geq 3$. In this paper, we study about $L(n, k)$ has a perfect matching or a near-perfect matching for all n and k and we find its matching number and edge cover number.

The perfect matching or the near-perfect matching of *L*(*n*, *k*) for each *n* and *k*

Definition 1. The *Union of a Kneser graph and a Johnson graph* denoted by $L(n, k)$ is the graph whose vertices are the k-element subsets of an n-element set, two vertices A and B are adjacent if and only if $A \cap B = \emptyset$ or $|A \cap B| = k - 1$.

Proposition 1. $\binom{n}{2}$ *is even if and only if n ≡ 0 or 1 (mod 4).*

Proof. (⇒) Assume $\binom{n}{2}$ is even.

$$\binom{n}{2} = \frac{n(n-1)}{2} = 2t \text{ for some integer } t.$$

Since $n(n-1) = 4t$, we have $4|n$ or $4|(n-1)$ or $2|n$ and $2|(n-1)$. So we have that $4|n$ or $4|(n-1)$ because n and $n-1$ are relatively prime.
Hence $n \equiv 0$ or 1 (mod 4).

(⇐) Assume $n \equiv 0$ or 1 (mod 4).
That is $4|n$ or $4|(n-1)$.
If $4|n$, we have $n = 4t_1$ for some integer t_1. Then

$$\begin{aligned}\binom{n}{2} &= \frac{n(n-1)}{2}\\ &= \frac{4t_1(n-1)}{2}\\ &= 2(t_1(n-1)).\end{aligned}$$

Thus $\binom{n}{2}$ is even.
If $4|(n-1)$, we have $n-1 = 4t_2$ for some integer t_2. Then

$$\begin{aligned}\binom{n}{2} &= \frac{n(n-1)}{2}\\ &= \frac{n(4t_2)}{2}\\ &= 2(nt_2).\end{aligned}$$

Thus $\binom{n}{2}$ is even. Therefore $\binom{n}{2}$ is even if and only if $n \equiv 0$ or 1 (mod 4).

Proposition 2. *If* $n \equiv 0$ *or* 1 (*mod 4*), *then* $L(n, 2)$ *has a perfect matching. Otherwise* $L(n, 2)$ *has a near-perfect matching.*

Proof. Suppose $n \equiv 0$ or 1 (*mod* 4). By Proposition 1, we have $L(n, 2)$ is an even complete graph. Thus $L(n, 2)$ has a perfect matching, otherwise $L(n, 2)$ is an odd complete graph, $L(n, 2)$ has a near-perfect matching.

Proposition 3. $J(n- 2)$ *has a perfect matching or a near-perfect matching.*

Proof. Case 1. $n = 2k$. We define a matching M as follows.
Let an edge joining $\{i, 2j\}$ and $\{i, 2j+1\}$ be in a matching M for each $i < 2j$; $i = 1, 2, ..., 2k-1$ and $j = 1, 2, ..., k - 1$ when i is odd.
Let an edge joining $\{I, 2j + 1\}$ and $\{I, 2j + 2\}$ be in a matching M for each $i < 2j + 1$;$i = 1,2, ..., 2k$ and $j = 1, 2, ..., k - 1$ when i is even.
The vertices that are not matched in M always have n as an element. So they induce a complete graph K_k. Thus we can find M' that is a perfect matching or a near-perfect matching in K_k. Thus $M \cap M'$ is a perfect matching or a near-perfect matching.

Case 2. $n = 2k + 1$. We can find a perfect matching or a near-perfect matching by a method similar to above case.

Corollary 1. $L(n, 2)$ *has a perfect matching or a near-perfect matching.*

Proposition 4. $\binom{n}{3}$ *is even if and only if n is even or* $n \equiv 1$ (*mod 4*) *for* $n \geq 4$.

Proposition 5. $J(n, 3)$ *has a perfect matching or a near-perfect matching.*

Proof. We prove by induction on n.

Base Case. It is straightforward to show that $J(3, 3)$ has a near-perfect matching and $J(4, 3)$ has a perfect matching.
Induction step. Let $n \geq 5$.
Consider $J(n, 3)$. Let

$$A = \{v \in V(J(n, 3)) \mid n \notin v\}$$
$$B = \{v \in V(J(n, 3)) \mid n \in v\}.$$

So $[V(A)] \cong J(n-1, 3)$ and $[V(B)] \cong J(n-1, 2)$. By induction hypothesis, $[V(A)]$ has a perfect matching or a near-perfect matching, say M_1.
By Lemma 3, $[V(B)]$ has a perfect matching or a near-perfect matching, say M_2.Then $M_1 \cup M_2$ is a matching in $J(n, 3)$.

Case 1. M_1 is a perfect matching in $[V(A)]$ and M_2 is a perfect matching in $[V(B)]$.
Thus $M_1 \cup M_2$ is a perfect matching in $J(n, 3)$.

Case 2. M_1 is a perfect matching in $[V(A)]$ and M_2 is a near-perfect matching in $[V(B)]$.
Thus $M_1 \cup M_2$ is a near-perfect matching in $J(n, 3)$.

Case 3. M_1 is a near-perfect matching in $[V(A)]$ and M_2 is a perfect matching in $[V(B)]$.
Thus $M_1 \cup M_2$ is a near-perfect matching in $J(n, 3)$.

Case 4. Without loss of generality, M_1 is a near-perfect matching in $[V(A)]$ that does not contain $\{1, 2, 3\}$ and M_2 is a near-perfect matching in $[V(B)]$ that does not contain $\{1; 2; n\}$. Let e be an edge incident to $\{1, 2, 3\}$ and $\{1, 2, n\}$.
Then $M_1 \cup M_2 \cup \{e\}$ is a perfect matching in $J(n, 3)$

Proposition 6. *Let* $n \geq 4$. *If* n *is even or* $n \equiv 1$ *(mod 4), then* $L(n, 3)$ *has a perfect matching. Otherwise* $L(n, 3)$ *has a near-perfect matching.*

Proof. It follows from Proposition 4 and 5.

Corollary 2. $L(n, 3)$ *has a perfect matching or a near-perfect matching.*

Theorem 1. $J(n, k)$ *has a perfect matching or a near-perfect matching.*

Proof. We prove by induction on n.

Base Case. $J(n, 1)$ and $J(k, k)$ are complete graphs, each of them has a perfect matching or has a near-perfect matching.
Induction step.
Consider $J(n, k)$. Let

$$A = \{v \in V(J(n, k)) \mid n \notin v\}$$
$$B = \{v \in V(J(n, k)) \mid n \in v\}.$$

So $[V(A)] \cong J(n-1, k)$ and $[V(B)] \cong J(n-1, k-1)$. By induction hypothesis, $[V(A)]$ has
a perfect matching or a near-perfect matching, say M_1.
$[V(B)]$ has a perfect matching or a near-perfect matching, say M_2. Then $M_1 \cup M_2$ is a matching in $J(n, k)$.

Case 1. M_1 is a perfect matching in $[V(A)]$ and M_2 is a perfect matching in $[V(B)]$. Thus $M_1 \cup M_2$ is a perfect matching in $J(n, k)$.

Case 2. M_1 is a perfect matching in $[V(A)]$ and M_2 is a near-perfect matching in $[V(B)]$.
Thus $M_1 \cup M_2$ is a near-perfect matching in $J(n, k)$.

Case 3. M_1 is a near-perfect matching in $[V(A)]$ and M_2 is a perfect matching in $[V(B)]$.
Thus $M_1 \cup M_2$ is a near-perfect matching in $J(n, k)$.

Case 4. Without loss of generality, M_1 is a near-perfect matching in $[V(A)]$ that does not contain $\{1, 2, \ldots, k\}$ and M_2 is a near-perfect matching in $[V(B)]$ that does not contain $\{1, 2, \ldots, k-1, n\}$. Let e be an edge incident to $\{1, 2, \ldots, k\}$ and $\{1, 2, \ldots, k-1, n\}$.

Then $M_1 \cup M_2 \cup \{e\}$ is a perfect matching in $J(n, k)$.

Corollary 3. *$L(n, k)$ has a perfect matching or a near-perfect matching.*

Proposition 7. *Let $n \geq 8$. $\binom{n}{4}$ is even if and only if $n = 8t, 8t+1, 8t+2$, or $8t+3$.*

Proposition 8. *$L(n, 4)$ has a perfect matching if and only if $n = 8t, 8t+1, 8t+2$, or $8t+3$. Otherwise $L(n, 4)$ has a near-perfect matching.*

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