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Semilattice Congruences on *E*-inversive Semigroups

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ABSTRACT

A congruence ρ on a semigroup S is a semilattice congruence on S if S/ρ is a semilattice. A semigroup S is called an E-inversive semigroup if for every $a\in S$ there is an element x in S such that ax is idempotent. In this paper, we investigated a semilattice congruence and an inverse congruence on E-inversive semigroups.

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INTRODUCTION

In 1955, Thierrin introduced the concept of an E-inversive semigroup. A semigroup S is called an *E-inversive semigroup* (Mitsch, 1990) if for every $a\in S$ there exists $x\in S$ such that ax is idempotent. Let $E(S)$ denote the set of all idempotents of a semigroup S. A semigroup S is called an *E-semigroup* (Weipoltshammer, 2002) if $E(S)$ forms a subsemigroup of S. A semigroup S is said to be a *band* if every element of S is idempotent, and a band S is rectangular (Clifford and Preston, 1961, p.10) if for all x, $y\in S$, $x = xyx$. A subsemigroup T of a semigroup is *normal* if $abcd = acbd$ for all a, b, c, $d\in T$. A commutative band is a *semilattice* (Clifford and Preston, 1961). An element a of a semigroup S is called *regular* if there exists x in S such that $a = axa$. A semigroup S is a *regular semigroup* (Howie, 1995) if all its elements are regular. A regular semigroup S is called an *inverse semigroup* (Howie, 1995) if its idempotents commute. For $a\in S$, $V(a) := \{x\in S \mid a = axa, x = xax\}$ is the set of all *inverse*s of a and $W(a) := \{x\in S \mid x = xax\}$ is the set of all *weak inverses* (Howie, 1995) of a. A congruence ρ on a semigroup S is called a *band congruence* (Petrich, 1973) if $(a, a^2)\in\rho$ for all $a\in S$ and a band congruence ρ on a semigroup S is called a *semilattice congruence* (Petrich, 1973) if $(ab, ba)\in\rho$ for all a, $b\in S$. A band congruence ρ is a *rectangular band congruence* if $(a, aba)\in\rho$ for all a, $b\in S$. Basic properties and results of E-inversive E-semigroup were given by Mitsch (1990), Zheng (1997) and Weipoltshammer (2002).

In this paper, we investigated characterizations of semilattice congruences on an E-inversive E-semigroup and an inverse congruence which we used full and weakly self-conjugate subsemigroups of a semigroup.

The following results are used in this research.

Lemma 1.1. (Weipoltshammer, 2002) A semigroup S is an E-inversive semigroup if and only if $W(a) \neq \varnothing$ for all $a\in S$.

Proposition 1.2. Let S be an E-inversive semigroup, and $a \in S$. If $x \in W(a)$ then $x \in W(axa)$, $axa \in W(x)$ and $xax \in W(a)$.

Proof. Let $x \in W(a)$. Then $x = xax = x(axa)x$. Therefore $x \in W(axa)$. Consider, $axa = a(xax)a = axaxaxa = (axa)x(axa)$ and $xax = xaxaxax = (xax)a(xax)$. Thus $axa \in W(x)$ and $xax \in W(a)$. □

Proposition 1.3. (Weipoltshammer, 2002) For any semigroup S, S is an E-semigroup if and only if $W(ab) = W(b)W(a)$ for all $a, b \in S$.

Proposition 1.4. (Weipoltshammer, 2002) Let S be an E-semigroup. Then

(i) for all $a \in S$, $a' \in W(a)$, $e, f \in E(S)$, $ea', a'f, fa'e \in W(a)$,

(ii) for all $a \in S$, $a' \in W(a)$, $e \in E(S)$, $a'ea, aea' \in E(S)$,

(iii) for all $e \in E(S)$, $W(e) \subseteq E(S)$,

(iv) for all $e, f \in E(S)$, $W(ef) = W(fe)$.

MAIN RESULTS

In this section, we find some special conditions for a semilattice congruence and an inverse congruence on E-inversive E-semigroups. Any semigroup S, the natural partial order (Mitsch, 1990) $\leq$ on S is defined by

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a = ay \text{ for some } x, y \in S^1.$$

For $a \in S$, if $a \geq e$ for some $e \in E(S)$ then $e = xa = ay$ and $ay \in E(S)$. A subset $E(a)$, $a \in S$, of an E-inversive semigroup S is definned by

$$E(a) := \{e \in E(S) \mid a \geq e\}.$$

Proposition 2.1. Let S be an E-inversive semigroup. A relation ρ on $E(S)$ is defined by $\rho := \{(a, b) \in E(S) \times E(S) \mid eaf = ebf \text{ for all } e, f \in E(S)\}$.

(i) If $E(S)$ is a rectangular band then ρ is a rectangular band congruence on $E(S)$.

(ii) If $E(S)$ is a normal band then ρ is a semilattice congruence on $E(S)$.

Proof. (i) Clearly, ρ is an equivalence relation on $E(S)$. Let $a, b, c \in E(S)$ be such that $a\rho b$.

Let $e, f \in E(S)$. Then $cf, ec \in E(S)$ since $E(S)$ is a regtangular band. By the definition of ρ, we have $eacf = ebcfc$ and $ecaf = ecbf$, it follows that $ac\rho bc$ and $ca\rho cb$. Thus ρ is a congruence on $E(S)$. For all $a, e, f \in E(S)$, $eaf = ea^2f$, so $a^2\rho a$. Since $E(S)$ is rectangular, $a\rho aba$ for all $a, b \in E(S)$. Hence ρ is a rectangular band congruence on $E(S)$.

(ii) If $E(S)$ is a normal band, then $eabf = ebaf$ for all $a, b, e, f \in E(S)$. Hence $ab\rho ba$ for all $a, b \in E(S)$. Therefore ρ is a semilattice congruence on $E(S)$.

□

Proposition 2.2. Let S be an E-inversive E-semigroup and let γ be a rectangular band congruence on $E(S)$. Then γ-class is a semilattice if and only if for all $e, f \in E(S)$, $e\gamma f$ if and only if $ef = fe$.

Proof. Suppose that γ-class is a semilattice. Let $e, f \in E(S)$ be such that $e\gamma f$. Then $e\gamma = f\gamma$. Note that $e \in e\gamma = f\gamma$ and $f \in f\gamma = e\gamma$, so $e, f \in e\gamma$. By assumption, we have $ef = fe$. On the other hand, let $e, f \in E(S)$ be such that $ef = fe$. Since $ef\gamma\,(ef)f\gamma\,(fe)f\gamma f$ and $ef\gamma fe\gamma\,(fe)e\gamma\,(ef)e\gamma e$, we have $e\gamma f$.

Clearly, if $e\gamma f$ if and only if $ef = fe$ for all $e, f \in E(S)$, then γ-class is a semilattice.

□

An E-inversive semigroup S is said to satisfy a condition ($*$) if

for all $x, y \in S$, $xy, yx \in E(S)$, implies $xy = yx$.

The following results satisfy a condition ($*$).

Lemma 2.3. Let S be an E-inversive semigroup satisfying a condition ($*$). If $ab = e$ and $e \in E(S)$ then $bea = e$.

Proof. Since $(bea)(bea) = b(eabe)a = b(eee)a = bea$, we have $bea \in E(S)$. Since $abe = ee = e$, we have $bea = abe = e$ by a condition ($*$).

□

Lemma 2.4. Let S be an E-inversive semigroup satisfying a condition ($*$). For all $a \in S$, $e \in E(S)$, $a \geq e$ if and only if $e \in S^1aS^1$.

Proof. Suppose that $a \geq e$, then there exist $x, y \in S^1$ such that $e = xa = ay$. Hence $e \in S^1aS^1$.

Suppose that $e \in S^1aS^1$, then there exist $x, y \in S^1$ such that $e = xay$. By Lemma 2.3, $a(yex) = e$ and $(yex)a = e$. Since $yex \in S^1$, we have $a \geq e$.

□

Theorem 2.5. If S is an E-inversive semigroup satisfying a condition ($*$) then a relation

$$\eta := \{(a, b) \in S \times S \mid E(a) = E(b)\}$$

is a semilattice congruence on S.

Proof. Clearly, η is an equivalence relation. We shall show that η is a compatible. Let $a, b, c \in S$ be such that $a\eta b$. Suppose that $e \in E(S)$ such that $ac \geq e$. Then there exists $x \in S^1$ such that $a(cx) = e$. By Lemma 2.3, $cxea = e$. Hence $a \geq e$. Since $E(a) = E(b)$, $b \geq e$ and so there exists $y \in S^1$ such that $yb = e$ and we have $(yb)(cxea) = e$. Note that $bc(xeaey) = e$ (by Lemma 2.3) and $(xeaeye)bc = e$ (by Lemma 2.3). Then $bc \geq e$ and so $E(ac) \subseteq E(bc)$. Similarly, we can show that $E(bc) \subseteq E(ac)$. Thus $E(ac) = E(bc)$ and $ac\eta bc$. The similar argument, we can show that $ca\eta cb$. Therefore η is a congruence on S.

To show that S/η is a band, let $a \in S$. If $a^2 \geq e$ then there exist $x, y \in S^1$ such that $e = a^2x = ya^2$, hence $e = a(ax) = (ya)a$ where $ax, ya \in S^1$ which implies that $a \geq e$, so $E(a^2) \subseteq \mathrm{E}(a)$.

Conversely, if $a \geq e$, then there exist $x, y \in S^1$ such that $e = xa = ay$. Thus $e = ee = (xa)(ay) = xa^2y$. Hence $e \in S^1a^2S^1$, so $a^2 \geq e$ by Lemma 2.4. Therefore $E(a^2) = E(a)$ and $a^2\eta a$.

Finally, we shall show that $ab\eta ba$ for all $a, b \in S$. Let $a, b \in S$. Suppose that $ab \geq e$. Then there exist $x, y \in S^1$ such that $e = abx = yab$. By Lemma 2.3, we obtain that

$$e = bxea = bx(yab)a = (bxya)ba$$

and

$$e = beya = b(abx)ya = ba(bxya).$$

Thus $ba \geq e$ and so $E(ab) \subseteq E(ba)$. Similarly, we can show that $E(ba) \subseteq E(ab)$, therefore $ab\eta ba$ and hence η is a semilattice congruence on S.

□

The last theorem, some conditions are given to find an inverse congruence on E-inversive semigroup. Recall that an inverse semigroup S is a regular semigroup in which every element of S has a unique inverse or S is a regular semigroup and its idempotents commute. On an orthodox semigroup S, the least inverse congruence γ is given by $\gamma = \{(a, b) \in S \times S \mid V(a) = V(b)\}$ (Hall, 1969). On an E-inversive E-semigroup, if we replace $V(a)$ and $V(b)$ by $W(aea')$ and $W(beb')$ respectively, then we obtain an inverse congruence on an E-inversive E-semigroup as follows:

Theorem 2.6. Let S be an E-inversive semigroup and let γ be a relation defined by

$\gamma := \{(a,b) \in S \times S \mid$ there exist $a' \in W(a)$, $b' \in W(b)$ such that $W(aea') = W(beb')$ for all $e \in E(S)\}$.

If $E(S)$ is a rectangular band then γ is an inverse congruence on S.

Proof. Since $E(S)$ is a rectangular band, S is an E-semigroup. Clearly, γ is reflexive and symmetric. We shall show that γ is transitive, let $a, b, c \in S$ be such that $a\gamma b$ and $b\gamma c$. Then there exist $a' \in W(a)$ and $b' \in W(b)$ such that $W(aea') = W(beb')$ for all $e \in E(S)$ and there exist $b^* \in W(b)$ and $c' \in W(c)$ such that $W(beb^*) = W(cec')$ for all $e \in E(S)$. Since $b', b^* \in W(b)$, by Proposition 1.4 (ii), we have $beb', beb^* \in E(S)$ for all $e \in E(S)$. By Proposition 1.4 (ii) again and $E(S)$ is a rectangular band, we have $W(beb') = W(b(eb^*be)b') = W((beb^*)(beb')) = W((beb')(beb^*)) = W(b(eb'be)b^*) = W(beb^*)$. Hence $W(aea') = W(cec')$ for all $e \in E(S)$ and so γ is transitive.

To show that γ is a compatible, let $a, b, c \in S$ be such that $a\gamma b$. Then there exist $a' \in W(a)$ and $b' \in W(b)$ such that $W(aea') = W(beb')$ for all $e \in E(S)$. For $e \in E(S)$, by Proposition 1.4 (ii), we have $cec' \in E(S)$ where $c' \in W(c)$. Then $W(a(cec')a') = W(b(cec')b')$ and $W[(ac)e(c'a')] = W[(bc)e(c'b')]$ where $c'a' \in W(ac)$ and $c'b' \in W(bc)$ by Proposition 1.3. Hence $ac\gamma bc$.

For $c \in S$, $c' \in W(c)$, $W(cec') = W(cec')$ for all $e \in E(S)$. For $e \in E(S)$, we have $aea', beb' \in E(S)$. Thus

$$\begin{aligned} W[(ca)e(a'c')] &= W[c(aea')c'] \\ &= W(c')W(aea')W(c) && \text{(by Proposition 1.3)} \\ &= W(c')W(beb')W(c) \\ &= W[c(beb')c'] && \text{(by Proposition 1.3)} \\ &= W[(cb)e(b'c')]. \end{aligned}$$

Note that $a'c' \in W(ca)$ and $b'c' \in W(cb)$. Therefore $ca\gamma cb$ and so γ is a congruence on S. To show that γ is a regular congruence on S, let $a \in S$. Then $a\gamma a$ and $W(aea') = W(aea')$ for all $e \in E(S)$, where $a' \in W(a)$. By Proposition 1.2, $a' \in W(aa'a)$ for all $a' \in W(a)$.

For $e \in E(S)$, $a' \in W(a)$ and by Proposition 1.4(ii), we have $aea' \in E(S)$. Consider,

$$\begin{aligned} W(aea') &= W[(aea')(aa')] \\ &= W[(aa')(aea')] && \text{(by Proposition 1.4(iv))} \end{aligned}$$

$$= W[(aa'a)ea].$$

Therefore $a\gamma(aa'a)$ and so γ is a regular congruence on S.

To show that γ is an inverse congruence on S, let $g, h \in E(S)$. Since S is an E-semigroup, by Proposition 1.4(iv), we have $W(gh) = W(hg)$ and $W(gh) = W(h)W(g)$. Consider

$$\begin{aligned} W[(gh)h(hg)] &= W(hg)W(h)W(gh) && \text{(by Proposition 1.3)} \\ &= W(gh)W(h)W(hg) \\ &= W[(hg)h(gh)]. \end{aligned}$$

Note that $hg \in W(gh)$ and $gh \in W(hg)$ by Proposition 1.3. Hence $gh\gamma hg$. Therefore γ is an inverse congruence on S.

□

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