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# Idempotency of linear combinations of commuting three tripotent matrices

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## ABSTRACT

Given nonzero commuting tripotent matrices $T_1, T_2$ and $T_3$, i.e., $T_i^3 = T_i$ and $T_iT_j = T_jT_i$, $i, j$ = 1, 2, 3, the problem of characterizing all situations, in which a linear combination $A = c_1T_1 + c_2T_2 + c_3T_3$ where $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$ is an idempotent matrix, is studied.

*Keywords* : idempotency, tripotent matrices

## INTRODUCTION

The symbols $\mathbb{C}$ and $M_n(\mathbb{C})$ are used to denote the sets of complex numbers and $n \times n$ complex matrices, respectively. It is assumed throughout that $a_0, a_1, a_2 \in \mathbb{C}$ are nonzero complex numbers and $T_0, T_1, T_2 \in M_n(\mathbb{C})$ are nonzero commuting tripotent complex matrices of order *n*, i.e., $T_i^3 = T_i$, and $T_iT_j = T_jT_i, i, j = 1, 2, 3.$ The purpose of this note is to characterize all situations in which a linear combination of $T_0, T_1$ and $T_2$ of the form

$$A = a_0T_0 + a_1T_1 + a_2T_2$$

is also an idempotent matrix. A similar problem, concerning the question of when a linear combination

$$T = c_1T_1 + c_2T_2$$

of nonzero tripotent matrices $T_1$ and $T_2 \in M_n(\mathbb{C})$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ is tripotent, has been solved by Baksalary *et al*. (2004). From their theorm it follows that the linear combination of tripotent $T = c_1T_1 + c_2T_2$ , where $T_1$ and $T_2$ are tripotent, is tripotent. Further results concerning the idempotency of linear combinations of matrices are given in (Baksalary and Baksalary, 2000; Baksalary *et al*., 2002). A very useful property of a tripotent matrix is that it can uniquely be represented as a difference of two idempotent matrices $B_1$ and $B_2$ which are disjoint in the sense that $B_1B_2 = 0$ and $B_2B_1 = 0$ (Baksalary, 2004)

## MAIN RESULTS

As already pointed out, the main result of this paper provides a complete solution to the problem of characterizing situations, in which a linear combination of three tripotent matrices is idempotent.

**Lemma 1.** (Baksalary *et al*., 2002). For nonzero $c_0, d_0 \in \mathbb{C}$ and nonzero tripotent matrices $T_0, T \in M_n(\mathbb{C})$ satisfying the commutativity property $T_0 T = TT_0$, let $A$ be their linear combination of the form $A = c_0 T_0 + d_0 T$ . Under the assumption that $T \neq T_0$ and $T_0 = -T$ , the matrix $A$ is tripotent if and only if one of the following conditions holds:

(a) $c_0 = 1,\ d_0 = -1$ or $c_0 = -1,\ d_0 = 1$ and $T_0^2 T = T_0 T^2$,

(b) $c_0 = 1,\ d_0 = -2$ or $c_0 = -1,\ d_0 = 2$ and $T_0^2 T = T = T_0 T^2$,

(c) $c_0 = 2,\ d_0 = -1$ or $c_0 = -2,\ d_0 = 1$ and $T_0^2 T = T_0 = T_0 T^2$,

(d) $c_0 = 1,\ d_0 = 1$ or $c_0 = -1,\ d_0 = -1$ and $T_0^2 T = -T_0 T^2$,

(e) $c_0 = 1,\ d_0 = 2$ or $c_0 = -1,\ d_0 = -2$ and $T_0^2 T = T = -T_0 T^2$,

(f) $c_0 = 2,\ d_0 = 1$ or $c_0 = -2,\ d_0 = -1$ and $T_0^2 T = -T_0 = -T_0 T^2$,

(g) $c_0 = \frac{1}{2},\ d_0 = \frac{1}{2}$ or $c_0 = \frac{1}{2},\ d_0 = -\frac{1}{2}$ or $c_0 = -\frac{1}{2},\ d_0 = \frac{1}{2}$ or $c_0 = -\frac{1}{2},\ d_0 = -\frac{1}{2}$ and $T_0^2 T = T,\ T_0 T^2 = T_0$.

From Lemma 1 we have equation $A = c_0 T_0 + d_0 T.$ Let $T$ be a linear combination $T = c_1 T_1 + c_2 T_2$ of nonzero tripotent matrices $T_1, T_2 \in M_n(\mathbb{C})$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ the combination of tripotent matrices $c_1 T_1 + c_2 T_2$ is also a tripotent matrix when $c_1, c_2$ follows as the condition (a) to (g) of Lemma 1. We will bring $T$ to replace in the linear combination of $A = c_0 T_0 + d_0 T$ and so we can get the equation as follows:

$$\begin{aligned} A &= c_0 T_0 + d_0 T \\ &= c_0 T_0 + d_0 (c_1 T_1 + c_2 T_2). \end{aligned} \tag{1}$$

By substituting $T = c_1 T_1 + c_2 T_2$ in the linear combination of the form $A = c_0 T_0 + d_0 T$, we have the following theorem.

**Theorem 1.** For nonzero $c_0, c_1, c_2, d_0 \in \mathbb{C}$, and nonzero tripotent matrices $T_0, T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property, let $A$ be their linear combination of the form

$$A = c_0 T_0 + d_0(c_1 T_1 + c_2 T_2).$$

Under the assumption that $T_0 \neq (c_1 T_1 + c_2 T_2)$ and $(c_1 T_1 + c_2 T_2) \neq -T_0$, the matrix $A$ is tripotent if and only if:

(a) $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$ or $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$ and $T_0^2(c_1 T_1 + c_2 T_2) = T_0(c_1 T_1 + c_2 T_2)^2$,

(b) $c_0 = 1, d_0 = -2, c_1 = 1, c_2 = -2$ or $c_0 = -1, d_0 = 2, c_1 = -1, c_2 = 2$ and $T_0^2(c_1 T_1 + c_2 T_2) = (c_1 T_1 + c_2 T_2) = T_0(c_1 T_1 + c_2 T_2)^2$,

(c) $c_0 = 2, d_0 = -1, c_1 = 2, c_2 = -1$ or $c_0 = -2, d_0 = 1, c_1 = -2, c_2 = 1$ and $T_0^2(c_1 T_1 + c_2 T_2) = T_0 = T_0(c_1 T_1 + c_2 T_2)^2$,

(d) $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$ or $c_0 = 1, d_0 = -1, c_1 = -1, c_2 = -1$ and $T_0^2(c_1 T_1 + c_2 T_2) = -T_0(c_1 T_1 + c_2 T_2)^2$,

(e) $c_0 = 1, d_0 = 2, c_1 = 1, c_2 = 2$ or $c_0 = -1, d_0 = -2, c_1 = -1, c_2 = -2$ and $T_0^2(c_1 T_1 + c_2 T_2) = (c_1 T_1 + c_2 T_2) = -T_0(c_1 T_1 + c_2 T_2)^2$,

(f) $c_0 = 2, d_0 = 1, c_1 = 2, c_2 = 1$ or $c_0 = -2, d_0 = -1, c_1 = -2, c_2 = -1$ and $T_0^2(c_1 T_1 + c_2 T_2) = -T_0 = -T_0(c_1 T_1 + c_2 T_2)^2$,

(g) $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$ and $T_0^2(c_1 T_1 + c_2 T_2) = (c_1 T_1 + c_2 T_2)$, $T_0(c_1 T_1 + c_2 T_2)^2 = T_0$.

**Theorem 2.** For nonzero $a_0, a_1, a_2 \in \mathbb{C}$, and nonzero tripotent matrices $T_0, T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property, let $A$ be their linear combination of the form

$$A = a_0 T_0 + a_1 T_1 + a_2 T_2$$

where $T_2 \neq T_1$ and $T_2 \neq -T_1$ Then the matrix $A$ is tripotent if and only if one of the following conditions holds:

(a) $a_0 = 1, a_1 = -1, a_2 = -1$ or $a_0 = -1, a_1 = -1, a_2 = 1$ and $T_0^2(T_1 - T_2) = T_0(T_1 - T_2)^2 = -T_0^2(T_1 - T_2)$,

(b) $a_0 = 1, a_1 = -2, a_2 = 4$ or $a_0 = -1, a_1 = -2, a_2 = 4$ and $T_0^2(T_1 - 2T_2) = (T_1 - 2T_2) = T_0(T_1 - 2T_2)^2$,

(c) $a_0 = 2, a_1 = -2, a_2 = 1$ or $a_0 = -2, a_1 = -2, a_2 = 1$ and
$T_0^2(T_2 - T_1) = T_0 = T_0(T_2 - T_1)^2$,

(d) $a_0 = 1, a_1 = 1, a_2 = 1$ or $a_0 = -1, a_1 = 1, a_2 = 1$ and $T_0^2(T_1 + T_2) = -T_0(T_1 + T_2)^2$,

(e) $a_0 = 1, a_1 = 2, a_2 = 4$ or $a_0 = -1, a_1 = 2, a_2 = 4$ and
$T_0^2(T_1 + T_2) = (T_1 + T_2) = -T_0(T_1 + T_2)^2$,

(f) $a_0 = 2, a_1 = 2, a_2 = 1$ or $a_0 = -2, a_1 = 2, a_2 = 1$ and
$T_0^2(2T_1 + T_2) = -T_0 = -T_0^2(2T_1 + T_2)$,

(g) $a_0 = \frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{4}$ or $a_0 = -\frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{4}$ and
$T_0^2(\frac{1}{2}T_1 + \frac{1}{2}T_2) = (\frac{1}{2}T_1 + \frac{1}{2}T_2)$, $T_0(\frac{1}{2}T_1 + \frac{1}{2}T_2)^2 = T_0$.

*Proof.* Let

$$A = c_0T_0 + d_0T \quad \text{where} \quad T = c_1T_1 + c_2T_2. \tag{3}$$

Direct calculations show that $A$ of the form (1) is tripotent if and only if

$$c_0^3T_0 + 3c_0^2d_0T_0^2T + 3c_0d_0^2T_0T^2 + d_0^3T = c_0T_0 + d_0T$$

or

$$(c_0^3 - c_0)T_0 + 3c_0^2d_0T_0^2T + 3c_0d_0^2T_0T^2 + (d_0^3 - d_0)T = 0. \tag{4}$$

Substituting $T = c_1T_1 + c_2T_2$ to (4) we have

$$(c_0^3 - c_0)T_0 + 3c_0^2d_0T_0^2(c_1T_1 + c_2T_2) + 3c_0d_0^2T_0(c_1T_1 + c_2T_2)^2 + (d_0^3 - d_0)(c_1T_1 + c_2T_2) = 0. \tag{5}$$

By (3), we can rewrite equation:

$$\begin{aligned} A &= c_0T_0 + d_0T \\ &= c_0T_0 + d_0(c_1T_1 + c_2T_2) \\ &= c_0T_0 + d_0c_1T_1 + d_0c_2T_2. \end{aligned}$$

Let $a_0 = c_0, a_1 = d_0c_1$ and $a_2 = d_0c_2$. Thus

$$A = a_0T_0 + a_1T_1 + a_2T_2 \tag{6}$$

By Theorem 1 together with (5) we consider the following case:

<u>Case (i)</u>. $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$ and $T_0^2(c_1T_1 + c_2T_2) = T_0(c_1T_1 + c_2T_2)^2$,

$$-3T_0^2(T_1 - T_2) + 3T_0(T_1 - T_2)^2 = 0,$$

$$-T_0^2(T_1 - T_2) + T_0(T_1 - T_2)^2 = 0,$$

$$-T_0^2(T_1 - T_2) + T_0^2(T_1 - T_2) = 0.$$

<u>Case (ii)</u>. $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$ and $T_0^2(c_1T_1 + c_2T_2) = T_0(c_1T_1 + c_2T_2)^2$,

$$3T_0^2(T_2 - T_1) - 3T_0(T_2 - T_1)^2 = 0,$$
$$T_0^2(T_2 - T_1) - T_0(T_2 - T_1)^2 = 0,$$
$$T_0^2(T_2 - T_1) - T_0^2(T_2 - T_1) = 0.$$

In this case, we have $a_0 = 1, a_1 = -1, a_2 = -1$ or $a_0 = -1, a_1 = -1, a_2 = 1$ and $T_0^2(T_1 - T_2) = T_0(T_1 - T_2)^2 = -T_0^2(T_1 - T_2)$.
From (b) to (g) the proofs are similar to (a).

By Baksalary (2004), the tripotent matrix $A$, in Theorem 2, can uniquely be represented as a difference of two idempotent matrices $A_1$ and $A_2$ which are disjoint in the sense that $A_1A_2 = 0$ and $A_2A_1 = 0$. Thus, $A = A_1 - A_2$ where $A_1$ and $A_2$ are idempotent matrices.

Given two different nonzero idempotent matrices $A_1$ and $A_2$, let $C$ be their linear combination of the form

$$C = c_1A_1 + (-c_2)A_2 \tag{7}$$

Direct calculations show that, in view of $C^2 = C$, a matrix $C$ of the form (7)

$$C = c_1A_1 + (-c_2)A_2$$
$$C^2 = c_1^2A_1^2 - 2c_1c_2A_1A_2 + c_2^2A_2^2$$
$$c_1A_1 + (-c_2)A_2 = c_1^2A_1^2 - 2c_1c_2A_1A_2 + c_2^2A_2^2$$
$$(c_1^2 - c_1)A_1^2 - 2c_1c_2A_1A_2 + (c_2^2 + c_2)A_2^2 = 0.$$

Then, in view of the Theorem (Baksalary *et al*., 2004), there is one case such that the matrix $C = c_1A_1 + c_2A_2$ (now equal to $C = c_1A_1 + (-c_2)A_2$) is idempotent where

$$c_1 = 1, c_2 = 1, \quad A_1A_2 = 0 = A_2A_1 \tag{8}$$

Hence $A$ is an idempotent matrix, under this condition, criterion (8). The proof is complete.

□

**Corollary 1**. For nonzero $c_0d_0 \in \mathbb{C}$ and nonzero tripotent matrices $T_0, T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property, let $A$ be their linear combination of the form

$$A = a_0T_0 + a_1T_1 + a_2T_2, \quad a_0, a_1, a_2 \in \mathbb{C},$$

where $T_i \neq T_j$ and $Ti \neq -Tj, \ \forall i, j = 1,2,3.$ Then:

(a) in case where $T_iT_j = 0, \forall i, j = 1,2,3.$ a matrix $A$ is tripotent if and only if $(a_0, a_1, a_2) \in \{(1,1,1), (1,-1,1), (-1,-1,1), (-1,1,1)\}$,

(b) in case $T_0T = T$, where $T = c_1T_1 + c_2T_2$ a matrix $A$ is tripotent if and only if $T$ is idempotent and $(a_0, a_1, a_2) \in \{(1,-1,1), (-1,-1,1), (-1,-2,4), (-1,-2,4)\}$, or $-T$ is idempotent and $(a_0, a_1, a_2) \in \{(1,1,1), (-1,1,1), (1,2,4), (-1,2,4)\}$, or $T_0$ is idempotent equal to $T$ and the pairs $(a_0, a_1, a_2)$ are as in Theorem 2(g),

(c) in case $T_0T = T_0$, where $T = c_1T_1 + c_2T_2$ a matrix *A* is tripotent if and only if $T_0$ is idempotent and $(a_0, a_1, a_2) \in \{(1,-1,1), (-1,-1,1), (2,-2,1), (-2,-2,1)\}$, or $-T_0$ is idempotent and $(a_0, a_1, a_2) \in \{(1,1,1), (-1,1,1), (2,2,1), (-2,2,1)\}$, or *T* is idempotent equal to $T_0^2$ and the pairs $(a_0, a_1, a_2)$ are as in Theorem 2(g).

*Proof.* It follows directly from Theorem 2. It seem interesting to show that *k* must be less than 3 when $c_1$ and $c_2$ are restricted to be real numbers.

□

**Theorem 3.** Let $c_1$ and $c_2$ be nonzero real numbers. Let *A* and *B* be nonzero complex matrices and $c_1A + c_2B = C$ satisfy $A^3 = A, B^{k+1} = B$, $AB = BA$, $A \neq B$, and $C^2 = C$. Then *B* is idempotent or tripotent matrix.

*Proof.* Let *A* be tripotent and *B* be *k*-potent. If $c_1A + c_2B$ is idempotent, then

$$\begin{aligned} c_1A + c_2B &= c_1(A_1 - A_2) + c_2B, && \text{(by [4, p. 22])} \\ &= (c_1A_1 - c_1A_2) + c_2B \\ &= (c_1A_1 + dA_2) + c_2B && (\text{where } d = c_1) \\ &= c_1A_1 + (dA_2 + c_2B). \end{aligned}$$

By Corollary (Benitez and Thome, 2005), *B* must be idempotent or tripotent.

If *B* is an idempotent matrix, then $c_1A_1 + (dA_2 + c_2B)$ is idempotent. From Theorem (Baksalary *et al*., 2004), asserts that there are $d, c_2$ such that $(dA_2 + c_2B)$ is idempotent. Now, let $E = (dA_2 + c_2B)$ be idempotent. From Theorem (Baksalary *et al*., 2004), we can find some scalar $k_1, k_2$ such that $k_1A_1 + k_2E$ is idempotent. Thus, there exist some scalars for which the combination of *A* and *B* is idempotent.

If *B* is a tripotent matrix, then $c_1A_1 + (dA_2 + c_2B)$ is idempotent. From Theorem (Baksalary, 2004), asserts that there are $d, c_2$ such that $(dA_2 + c_2B)$ is idempotent. Now, let $Q = (dA_2 + c_2B)$ be idempotent. From Theorem (Baksalary *et al*., 2004), we can find some scalar $l_1, l_2$ such that $l_1A_1 + l_2Q$ is idempotent. Thus, there exist some scalars for which the combination of *A* and *B* is idempotent.

□

**Corollary 2.** Let *A*, *B* and *C* be nonzero complex matrices. If one of *A* or *B* or *C* is idempotent (or tripotent) and the combination *A*, *B* and *C* is idempotent (or tripotent), then the others matrices must be idempotent (or tripotent).

*Proof.* It follows directly from Theorem 3.

□

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