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A General Iterative Method for Nonexpansive Mappings

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ABSTRACT

Let H be a real Hilbert space and $T: H \to H$ be a nonexpansive mapping, $f: H \to H$ a contraction mapping with coefficient $0 < \alpha < 1$, A a strongly positive bounded linear operator with coefficient $\tilde{\gamma} > 0$, and $0 < \gamma < \tilde{\gamma}/\alpha$. It is proved that both sequences $\{x_n\}$ and $\{w_n\}$ generated by the iterative method $x_n = \alpha_n\gamma f(x_n) + (I - (\alpha_n + \beta_n) A)Tx_n + \beta_n u_n$, and $w_{n+1} = \alpha_n\gamma f(w_n) + (I - (\alpha_n + \beta_n) A)Tw_n + \beta_n u_n$ converge strongly to a fixed point $\tilde{x} \in F(T)$ which solves the variational inequality $\langle (A - \gamma f)\tilde{x}, \tilde{x} - x\rangle \leq 0$ for $x \in F(T)$. Our results extend and improve the corresponding results of G. Marino and H.K. Xu [A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318(2006), 43-52], and may others.

Keywords: Nonexpansive mapping, Iterative method, Variational inequality, Fixed point; Viscosity approximation

INTRODUCTION

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., (Deutsch and Yamada, 1998; Xu, 2002; Xu, 2003; Yamada, 2001; Yamada, Ogura, Yamashita, and Sakaniwa, 1998) and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. Assume A is strongly positive; that is, there is a constant $\tilde{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \tilde{\gamma}\|x\|^2 \qquad \text{for all } x \in H. \tag{2}$$

Recall that $T: H \to H$ is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is the set $F(T) := \{x \in H : Tx = x\}$. We assume that $C = F(T)$. It is well known that $F(T)$ is closed convex (Geobel and Kirk, 1990). In (Xu, 2003) [see also (Yamada, 2001)], it is proved that the sequence

$\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \qquad n \geq 0, \tag{3}$$

converges strongly to the unique solution of the minimization problem (1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions that will be made precise in Section 3.

On the other hand, Moudafi (2000) introduced the viscosity approximation method for nonexpansive mappings (Xu, 2004) for further developments in both Hilbert and Banach spaces. Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \sigma_n)Tx_n + \sigma_n f(x_n), \qquad n \geq 0, \tag{4}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved (Moudafi, 2000; Xu, 2004) that under a certain appropriate condition imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (4) strongly converges to the unique solution x^* in C of the variational inequality

$$\left\langle (I - f)x^*, x - x^* \right\rangle \geq 0, \qquad x \in C. \tag{5}$$

In 2006, Marino and Xu combined the iterative method (3) with the viscosity approximation method (4) and consider the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \qquad n \geq 0. \tag{6}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution of the variational inequality

$$\left\langle (A - \gamma f)x^*, x - x^* \right\rangle \geq 0, \qquad x \in C, \tag{7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h\langle x\rangle,$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

The main purpose of this paper is to consider the following iteration in a Hilbert space:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - (\alpha_n + \beta_n)A)Tx_n + \beta_n u_n, \qquad n \geq 0. \tag{8}$$

We will prove that if the sequence $\{\alpha_n\}$, $\{\beta_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (8) converges strongly to the unique solution of the variational inequality

$$\left\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \right\rangle \leq 0, \qquad z \in F(T). \tag{9}$$

PRELIMINARIES

This section collects some lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Lemma 1 (Xu, 2002). Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} = (I-\gamma_n)a_n + \delta_n, \qquad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(*i*) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 (Wang, 1991). Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following conditions:

$$a_{n+1} = (1-\lambda_n)a_n + b_n + c_n \qquad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n/\lambda_n \to 0$, and $\sum_{n=0}^{\infty} c_n < \infty$ then $\alpha_n \to 0$ (as $n \to \infty$).

Lemma 3 (Geobel and Kirk, 1990). Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \varnothing$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.

The following lemma is not hard to prove.

Lemma 4 (Marino and Xu, 2006). Let H be a Hilbert space, K a closed convex subset of H, and $f : H \to H$ a contraction with coefficient $0 < \alpha < 1$, and A a strongly positive linear bounded operator with coefficient $\tilde{\gamma} > 0$. Then, for $0 < \gamma < \tilde{\gamma}/\alpha$,

$$\langle x-y, (A-\gamma f)x - (A-\gamma f)y \rangle \geq (\tilde{\gamma} - \gamma\alpha)\|x-y\|^2, \qquad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\tilde{\gamma} - \gamma\alpha$.

Recall the metric (nearest point) projection P_K from a real Hilbert space H to a closed convex subset K of H is defined as follows: given $x \in H$, $P_K x$ is the only point in K with the property $\|x - P_K x\| = \inf\{\|x-y\| : y \in K\}$.

P_K is characterized as follows.

Lemma 5 (Marino and Xu, 2006). Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \qquad \forall z \in K.$$

Lemma 6 (Marino and Xu, 2006). Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\tilde{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\tilde{\gamma}$.

Notation. We use $\rightarrow$ for strong convergence and $\xrightarrow{w}$ for weak convergence.

MAIN RESULTS

Let H be a real Hilbert space, A a bounded linear operator on H, and T a nonexpansive mapping on H (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$). Assume the set $F(T)$ is closed convex, the nearest point projection from H onto $F(T)$ is well defined.

Throughout the rest of this paper, we always assume that A is strongly positive; that is, there is a constant $\tilde{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \tilde{\gamma}\|x\|^2 \qquad \text{for all } x \in H. \tag{10}$$

(Note: $\tilde{\gamma} > 0$ is throughout reserved to be the constant such that (10) holds.)

Recall also that a contraction on H is a self-mapping f of H such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\| \qquad x, y \in H,$$

where $\alpha \in [0,1)$ is a constant.

Denote by $\prod$ the collection of all contraction on H; namely,

$$\prod = \{ f : f \text{ a contraction on } H \}.$$

Now given $f \in \prod$ with contraction coefficient $0 < \alpha < 1$, $0 < \gamma < \tilde{\gamma}/\alpha$ and $\{u_n\}$ is bounded sequence in H. Consider a mapping S_n on H defined by

$$S_n x = \alpha_n \gamma f(x) + (I - (\alpha_n + \beta_n)A)Tx + \beta_n u_n \quad \text{for all } x \in H. \tag{11}$$

It is easy to see that S_n is a contraction. Indeed, by Lemma 6, we have:

$$\begin{aligned} \|S_n x - S_n y\| &\leq \alpha_n \gamma \|f(x) - f(y)\| + \|I - (\alpha_n + \beta_n)\|\|Tx - Ty\| \\ &\leq (1 - (\tilde{\gamma} - \gamma\alpha)\alpha_n)\|x - y\|. \end{aligned}$$

Hence S_n has a unique fixed point, denoted x_n, which uniquely solves the fixed point equation

$$x_n = \alpha_n \gamma f(x_n) + (I - (\alpha_n + \beta_n)A)Tx_n + \beta_n u_n. \tag{12}$$

Note that x_n indeed depends on f as well, but we will suppress this dependence of x_n on f for simplicity of notation throughout the rest of this paper. We will also always use γ to mean a number in $(0, \tilde{\gamma}/\alpha)$.

The next proposition summarizes the basic properties of $\{x_n\}$.

Proposition 7. Let x_n be defined by (12).

(i) If $\{u_n\}$ is a bounded sequence then $\{x_n\}$ is bounded for $\alpha_n \in (0, \|A\|^{-1})$.

(ii) If $\alpha_n \to 0,\ \beta_n \to 0,$ and $\{u_n\}$ is a bounded sequence then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$

Proof. First observe that for $\alpha_n \in (0, \|A\|^{-1}),$ therefore $\|I - \alpha_n A\| \le 1 - \alpha_n \tilde{\gamma}$ by Lemma 6.

To show (i) pick a $p \in F(T).$ Then, by Lemma 6, we have

$$\begin{aligned}
\|x_n - p\| &= \|(I - (\alpha_n + \beta_n)A)(Tx_n - p) + \alpha_n(\gamma f(x_n) - Ap) - \beta_n(Ap - u_n)\| \\
&\le \|I - (\alpha_n + \beta_n)A\|\|Tx_n - p\| + \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|Ap - u_n\| \\
&\le (1 - (\alpha_n + \beta_n)\tilde{\gamma})\|x_n - p\| + \alpha_n\|\gamma f(x_n) - f(p) + f(p) - Ap\| \\
&\quad + \beta_n\|Ap - u_n\| \\
&\le (1 - (\alpha_n + \beta_n)\tilde{\gamma})\|x_n - p\| + \alpha_n\left[\alpha\gamma\|x_n - p\| + \|\gamma f(p) - Ap\|\right] \\
&\quad + \beta_n\|Ap - u_n\| \\
&\le (1 - (\tilde{\gamma} - \alpha\gamma)\alpha_n)\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|Ap - u_n\|.
\end{aligned}$$

It follows that

$$\|x_n - p\| \le \frac{1}{\tilde{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\| + \frac{\beta_n}{\tilde{\gamma} - \gamma\alpha}\|Ap - u_n\|,$$

for all $n \in \mathbb{N}$.

Hence $\{x_n\}$ is bounded, and therefore $\{f(x_n)\}$ and $\{ATx_n\}$ are also bounded.

(ii) Since $\alpha_n \to 0$ and $\beta_n \to 0,$ it follows that

$$\begin{aligned}
\|x_n - Tx_n\| &= \|\alpha_n\gamma f(x_n) + (I - (\alpha_n + \beta_n)A)Tx_n + \beta_n u_n - Tx_n\| \\
&= \|\alpha_n\gamma f(x_n) - (\alpha_n + \beta_n)ATx_n + \beta_n u_n\| \\
&\le \alpha_n\|\gamma f(x_n) - ATx_n\| + \beta_n\|ATx_n - u_n\| \to 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

■

Our first main result below shows that $\{x_n\}$ converges strongly as $n \to \infty$ to a fixed point of T which solves some variational inequality.

Theorem 8. Let $\{\alpha_n\} \in (0, \|A\|^{-1}),\ \alpha_n \to 0,\ \beta_n \to 0,\ \{u_n\}$ be a bounded sequence, and $\{x_n\}$ defined by (12). Then $\{x_n\}$ converges strongly as $n \to 0$ to a fixed point $\tilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \le 0, \qquad z \in F(T). \tag{13}$$

Equivalently, we have $P_{F(T)}\left(I - A + \gamma f\right)\tilde{x} = \tilde{x}.$

Proof. Firstly, we note by $\{x_n\}$ is a bounded sequence in a Hilbert space H, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \xrightarrow{w} x^*$. By Lemma 3, we have $x^* \in F(T)$. Next, we show that the solution of the variational inequality (13) is unique. Suppose $\tilde{x} \in F(T)$ and $\hat{x} \in F(T)$ both are solutions to (13); then

$$\langle (A-\gamma f)\tilde{x}, \tilde{x}-\hat{x} \rangle \leq 0 \tag{14}$$

and

$$\langle (A-\gamma f)\hat{x}, \hat{x}-\tilde{x} \rangle \leq 0. \tag{15}$$

Adding up (14) and (15), we obtain

$$\langle (A-\gamma f)\tilde{x}-(A-\gamma f)\hat{x}, \tilde{x}-\hat{x} \rangle \leq 0. \tag{16}$$

The strong monotonicity of $A-\gamma f$ (Lemma 4) implies that $\tilde{x}= \hat{x}$ and the uniqueness is proved. Below we use $\tilde{x} \in F(T)$ to denote the unique solution of (13). Now, we show $x_{n_j} \to x^*$. Let $z \in F(T)$, consider

$$x_n - z = \alpha_n(\gamma f(x_n) - Az) + (I-(\alpha_n+\beta_n)A)(Tx_n - z) - \beta_n(Az-u_n).$$

Thus, we have

$$\begin{aligned} \|x_n - z\|^2 &= \alpha_n \langle \gamma f(x_n) - Az, x_n - z\rangle + \langle (I-(\alpha_n+\beta_n)A)(Tx_n-z), x_n - z\rangle \\ &\quad - \beta_n \langle Az-u_n, x_n - z\rangle \\ &\leq (1-(\alpha_n+\beta_n)\tilde{\gamma})\|x_n - z\|^2 + \alpha_n \langle \gamma f(x_n) - Az, x_n - z\rangle \\ &\quad - \beta_n \langle Az-u_n, x_n - z\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - z\|^2 &\leq \frac{\alpha_n}{(\alpha_n+\beta_n)\tilde{\gamma}} \langle \gamma f(x_n) - Az, x_n - z\rangle - \frac{\beta_n}{(\alpha_n+\beta_n)\tilde{\gamma}} \langle Az-u_n, x_n - z\rangle \\ &= \frac{\alpha_n}{(\alpha_n+\beta_n)\tilde{\gamma}} \left[\gamma \langle f(x_n) - f(z), x_n - z\rangle + \langle \gamma f(z) - Az, x_n - z\rangle \right] \\ &\quad - \frac{\beta_n}{(\alpha_n+\beta_n)\tilde{\gamma}} \langle Az-u_n, x_n - z\rangle \\ &\leq \frac{\alpha_n}{(\alpha_n+\beta_n)\tilde{\gamma}} \left[\gamma\alpha \|x_n - z\|^2 + \langle \gamma f(z) - Az, x_n - z\rangle \right] \\ &\quad - \frac{\beta_n}{(\alpha_n+\beta_n)\tilde{\gamma}} \langle Az-u_n, x_n - z\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_n - z\|^2 &\leq \frac{\alpha_n}{(\alpha_n+\beta_n)\tilde{\gamma} - \alpha\alpha_n\gamma} \langle \gamma f(z) - Az, x_n - z\rangle \\ &\quad - \frac{\beta_n}{(\alpha_n+\beta_n)\tilde{\gamma} - \alpha\alpha_n\gamma} \langle Az-u_n, x_n - z\rangle. \end{aligned}$$

At this point, without loss of generality we may assume that $\langle Az-u_n, x_n-z\rangle < 0.$ So we have $\langle u_n-Az, x_n-z\rangle > 0.$ It follows that

$$\|x_n-z\|^2 \le \frac{1}{\tilde{\gamma}-\alpha\gamma}\langle \gamma f(z)-Az, x_n-z\rangle \to 0. \tag{17}$$

Thus, $x_{n_j} \to z$ as $j\to\infty$, since $x_{n_j} \xrightarrow{w} z.$

We next prove that x^* solves the variation inequality (13). Since

$$x_n = \alpha_n\gamma f(x_n)+(I-(\alpha_n+\beta_n)A)Tx_n+\beta_n u_n, \tag{18}$$

we derive, that

$$(A-\gamma f)x_n = -\frac{1}{\alpha_n}\left[(I-\alpha_n A)(I-T)x_n+\beta_n(ATx_n-u_n)\right].$$

It follows by the monotonicity of $I-T$ for $z\in F(T)$,

$$\begin{aligned}\langle (A-\gamma f)x_n, x_n-z\rangle &= -\frac{1}{\alpha_n}\langle (I-\alpha_n A)(I-T)x_n+\beta_n(ATx_n-u_n), x_n-z\rangle\\ &= -\frac{1}{\alpha_n}\langle (I-T)x_n-(I-T)z, x_n-z\rangle+\langle A(I-T)x_n, x_n-z\rangle\\ &\quad -\frac{\beta_n}{\alpha_n}\langle ATx_n-u_n, x_n-z\rangle\\ &\le \langle A(I-T)x_n, x_n-z\rangle-\frac{\beta_n}{\alpha_n}\langle ATx_n-u_n, x_n-z\rangle.\end{aligned}$$

At this point, without loss of generality we may assume that $\langle ATx_n-u_n, x_n-z\rangle < 0.$
So we have $\langle u_n-ATx_n, x_n-z\rangle > 0.$ It follows that

$$\langle (A-\gamma f)x_n, x_n-z\rangle \le \|A\|\|x_n-Tx_n\|\|x_n-z\| \to 0, \tag{19}$$

since $\|x_n-Tx_n\|\to 0.$ In particular, we consider a subsequence $\{x_{n_j}\}\subset\{x_n\}$ to obtain,

$$\langle (A-\gamma f)x^*, x^*-z\rangle \le 0.$$

That is $x^*\in F(T)$ is a solution of (13); hence $x^*=\tilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_n\}$ (as $n\to 0$) equals $\tilde{x}.$ Therefore, $x_n\to\tilde{x}$ as $n\to 0.$

The variational inequality (13) can be rewritten as

$$\langle [(I-A+\gamma f)\tilde{x}]-\tilde{x}, \tilde{x}-z\rangle \ge 0, \qquad z\in F(T).$$

This, by Lemma 5, is equivalent to the fixed point equation

$$P_{F(T)}\left(I-A+\gamma f\right)\tilde{x} = \tilde{x}.$$

∎

If $\{\beta_n\} \equiv 0$ in Theorem 8, we get,

Corollary 9. (Marino and Xu, 2006). Let $z_n \in H$ be the unique fixed point of the contraction $z \mapsto \alpha_n \gamma f(z) + (I - \alpha_n A)Tz.$ Then $\{z_n\}$ converges strongly as $n \to 0$ to the unique solution $\tilde{z} \in F(T)$ of the variational inequality

$$\langle (A - \gamma f)\tilde{z}, \tilde{z} - z \rangle \leq 0, \qquad z \in F(T).$$

Next we study a general iterative method as follows. The initial guess w_0 is selected in H arbitrarily, and the $(n+1)^{th}$ iterate w_{n+1} is recursively defined by

$$w_{n+1} = \alpha_n \gamma f(w_n) + (I - (\alpha_n + \beta_n)A)Tw_n + \beta_n u_n, \quad n \geq 0, \tag{20}$$

where $\{\alpha_n\} \in (0, \|A\|^{-1}),$ $\{\beta_n\} \subseteq (0,1),$ and $\{\alpha_n + \beta_n\} \subseteq [0,1)$ for all $n \in \mathbb{N},$ satisfying the following conditions:

$(C1)$ $\alpha_n \to 0;$

$(C2)$ $\sum_{n=0}^{\infty} \alpha_n = \infty;$

$(C3)$ either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \dfrac{\alpha_{n+1}}{\alpha_n} = 1.$

Below is the second main result of this paper.

Theorem 10. Let $\{w_n\}$ be generated by algorithm (20) with the sequence $\{\alpha_n\}$ of parameters satisfying conditions $(C1)-(C3),$ $\{u_n\}$ a bounded sequence, $\sum_{n=1}^{\infty} \beta_n < \infty,$ and $\lim_{n\to\infty} \dfrac{\beta_{n-1}}{\alpha_n} = 0.$ Then $\{w_n\}$ converges strongly to that is obtained Theorem 8.

Proof. Since $\alpha_n \to 0$ by condition $(C1),$ we may assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n.$ We now observe that $\{w_n\}$ is bounded. Indeed, pick any $p \in F(T)$ to obtain

$$\begin{aligned}
\|w_{n+1} - p\| &= \|(I - (\alpha_n + \beta_n)A)(Tw_n - p) + \alpha_n(\gamma f(w_n) - Ap) + \beta_n(u_n - Ap)\| \\
&\leq \|I - (\alpha_n + \beta_n)A\|\|Tw_n - p\| + \alpha_n\|\gamma f(w_n) - Ap\| + \beta_n\|u_n - Ap\| \\
&\leq (1 - (\alpha_n + \beta_n)\tilde{\gamma})\|w_n - p\| + \alpha_n\left[\|\gamma f(w_n) - f(p)\| + \|\gamma f(p) - Ap\|\right] \\
&\quad + \beta_n\|u_n - Ap\| \\
&\leq (1 - \alpha_n\tilde{\gamma} + \alpha\alpha_n\gamma + \beta_n\tilde{\gamma})\|w_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|u_n - Ap\| \\
&= (1 - (\tilde{\gamma} - \alpha\gamma)\alpha_n - \beta_n\tilde{\gamma})\|w_n - p\| + (\tilde{\gamma} - \alpha\gamma)\alpha_n \frac{\|\gamma f(p) - Ap\|}{\tilde{\gamma} - \alpha\gamma} \\
&\quad + \beta_n\tilde{\gamma}\frac{\|u_n - Ap\|}{\tilde{\gamma}}.
\end{aligned}$$

It follows from induction that

$$\|w_n - p\| \leq \max\left\{\|w_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\tilde{\gamma} - \alpha\gamma}, K\right\},$$

where

$$K \leq \sup\left\{\frac{\|u_n - Ap\|}{\tilde{\gamma}} : n \in \mathbb{N}\right\}.$$

As a result, noticing $\|w_{n+1} - Tw_n\| \leq \alpha_n \|\gamma f(w_n) - ATw_n\| - \beta_n \|ATw_n - u_n\|$, $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and $\{u_n\}$ is a bounded sequence, we obtain

$$\|w_{n+1} - Tw_n\| \to 0. \tag{21}$$

Now, we show

$$\|w_{n+1} - w_n\| \to 0. \tag{22}$$

Consider,

$$\begin{aligned}
\|w_{n+1} - w_n\| &= \|(I - (\alpha_n + \beta_n)A)(Tw_n - Tw_{n-1}) + \alpha_n\gamma(f(w_n) - f(w_{n-1})) \\
&\quad + (\alpha_n - \alpha_{n-1})(\gamma f(w_{n-1}) - ATw_{n-1}) + (\beta_n - \beta_{n-1})(u_{n-1} - ATw_{n-1}) \\
&\quad + \beta_n(u_n - u_{n-1})\| \\
&\leq (1 - (\alpha_n + \beta_n)\tilde{\gamma})\|w_n - w_{n-1}\| + \alpha\alpha_n\gamma\|w_n - w_{n-1}\| + M'|\alpha_n - \alpha_{n-1}| \\
&\quad + M'|\beta_n - \beta_{n-1}| + M'|\beta_n| \\
&\leq (1 - (\tilde{\gamma} - \alpha\gamma)\alpha_n)\|w_n - w_{n-1}\| + M'\left(|\alpha_n - \alpha_{n-1}| + 2|\beta_n| + |\beta_{n-1}|\right),
\end{aligned} \tag{23}$$

where $M' \leq \sup\left\{\max\left\{\|ATw_n\|, \|f(w_n)\|\right\} : n \geq 0\right\} < \infty$.

By assumption, we note that $\sum_{n=1}^{\infty}\left(\tilde{\gamma} - \alpha\gamma\right)\alpha_n = \infty$ and $\sum_{n=1}^{\infty} M'\left(|\alpha_n - \alpha_{n-1}| + 2|\beta_n| + |\beta_{n-1}|\right) < \infty$.

Hence by Lemma 1, we have $\|w_n - w_{n+1}\| \to 0$.

We now show that

$$\|w_n - Tw_n\| \to 0. \tag{24}$$

Indeed this follows from (22)

$$\begin{aligned}
\|w_n - Tw_n\| &= \|w_n - w_{n+1} + w_{n+1} - Tw_n\| \\
&\leq \|w_n - w_{n+1}\| + \|\alpha_n\gamma f(w_n) + (I - (\alpha_n + \beta_n)A)Tw_n + \beta_n u_n - Tw_n\| \\
&= \|w_n - w_{n+1}\| + \|\alpha_n\gamma f(w_n) - (\alpha_n + \beta_n)ATw_n + \beta_n u_n\| \\
&\leq \|w_n - w_{n+1}\| + \alpha_n\|\gamma f(w_n) - ATw_n\| + \beta_n\|ATw_n - u_n\| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

We next show that

$$\limsup_{n\to\infty}\left\langle Tw_n - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\right\rangle \leq 0, \tag{25}$$

where $\tilde{w}$ is obtained in Theorem 8. To see this, we take a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\left\langle w_n - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\right\rangle = \lim_{j\to\infty}\left\langle w_{n_j} - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\right\rangle.$$

Since $\{w_{n_j}\}$ is bounded, there exists a subsequence $\{w_{n_k}\}$ of $\{w_{n_j}\}$ such that $w_{n_k} \xrightarrow{w} z$ and

$$\limsup_{n\to\infty}\langle w_n - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\rangle = \lim_{k\to\infty}\langle w_{n_k} - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\rangle.$$

It follows from Lemma 3 and (24) that $z \in F(T)$. Hence by the variational inequality (13), we obtain

$$\limsup_{n\to\infty}\langle w_n - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\rangle = \langle z - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\rangle \le 0.$$

So (25) holds, by (24). Finally, we prove $w_n \to \tilde{w}$. To this end, we calculate

$$\begin{aligned}
\|w_{n+1} - \tilde{w}\|^2 &= \|(I - (\alpha_n + \beta_n)A)(Tw_n - \tilde{w}) + \alpha_n(\gamma f(w_n) - A\tilde{w}) + \beta_n(u_n - A\tilde{w})\|^2 \\
&= \|I - (\alpha_n + \beta_n)A\|^2 \|Tw_n - \tilde{w}\|^2 + \alpha_n^2\|\gamma f(w_n) - A\tilde{w}\|^2 + \beta_n^2\|u_n - A\tilde{w}\|^2 \\
&\quad + 2\alpha_n\langle (I - (\alpha_n + \beta_n)A)(Tw_n - \tilde{w}), \gamma f(w_n) - A\tilde{w}\rangle \\
&\quad + 2\beta_n\langle (I - (\alpha_n + \beta_n)A)(Tw_n - \tilde{w}), u_n - A\tilde{w}\rangle \\
&\quad + 2\alpha_n\beta_n\langle \gamma f(w_n) - A\tilde{w}, u_n - A\tilde{w}\rangle \\
&\le (1 - (\alpha_n + \beta_n)\tilde{\gamma})^2\|w_n - \tilde{w}\|^2 + \alpha_n^2\|\gamma f(w_n) - A\tilde{w}\|^2 + \beta_n^2\|u_n - A\tilde{w}\|^2 \\
&\quad + 2\alpha_n\langle Tw_n - \tilde{w}, \gamma f(w_n) - A\tilde{w}\rangle \\
&\quad - 2\alpha_n(\alpha_n + \beta_n)\langle A(Tw_n - \tilde{w}), \gamma f(w_n) - A\tilde{w}\rangle \\
&\quad + 2\beta_n\langle Tw_n - \tilde{w}, u_n - A\tilde{w}\rangle - 2\beta_n(\alpha_n + \beta_n)\langle A(Tw_n - \tilde{w}), u_n - A\tilde{w}\rangle \\
&\quad + 2\alpha_n\beta_n\langle \gamma f(w_n) - A\tilde{w}, u_n - A\tilde{w}\rangle \\
&= (1 - (\alpha_n + \beta_n)\tilde{\gamma})^2\|w_n - \tilde{w}\|^2 + \alpha_n^2\|\gamma f(w_n) - A\tilde{w}\|^2 + \beta_n^2\|u_n - A\tilde{w}\|^2 \\
&\quad + 2\alpha_n\langle Tw_n - T\tilde{w}, \gamma(f(w_n) - f(\tilde{w}))\rangle + 2\alpha_n\langle Tw_n - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\rangle \\
&\quad - 2\alpha_n^2\langle A(Tw_n - \tilde{w}), \gamma f(w_n) - A\tilde{w}\rangle + 2\beta_n\langle Tw_n - \tilde{w}, u_n - A\tilde{w}\rangle \\
&\quad - 2\beta_n^2\langle A(Tw_n - \tilde{w}), u_n - A\tilde{w}\rangle + 2\alpha_n\beta_n\langle \gamma f(w_n) - A\tilde{w}, u_n - A\tilde{w}\rangle \\
&\quad - 2\alpha_n\beta_n\langle A(Tw_n - \tilde{w}), \gamma f(w_n) - A\tilde{w}\rangle \\
&\quad - 2\alpha_n\beta_n\langle A(Tw_n - \tilde{w}), u_n - A\tilde{w}\rangle \\
&\le \left[(1 - (\alpha_n + \beta_n)\tilde{\gamma})^2 + 2\alpha\alpha_n\gamma\right]\|w_n - \tilde{w}\|^2 + \alpha_n^2\|\gamma f(w_n) - A\tilde{w}\|^2 \\
&\quad + \beta_n^2\|u_n - A\tilde{w}\|^2 + 2\alpha_n\langle Tw_n - \tilde{w}, \gamma f(\tilde{w}) - A\tilde{w}\rangle \\
&\quad - 2\alpha_n^2\langle A(Tw_n - \tilde{w}), \gamma f(w_n) - A\tilde{w}\rangle + 2\beta_n\langle Tw_n - \tilde{w}, u_n - A\tilde{w}\rangle \\
&\quad - 2\beta_n^2\langle A(Tw_n - \tilde{w}), u_n - A\tilde{w}\rangle + 2\alpha_n\beta_n\langle \gamma f(w_n) - A\tilde{w}, u_n - A\tilde{w}\rangle \\
&\quad - 2\alpha_n\beta_n\langle A(Tw_n - \tilde{w}), \gamma f(w_n) - A\tilde{w}\rangle \\
&\quad - 2\alpha_n\beta_n\langle A(Tw_n - \tilde{w}), u_n - A\tilde{w}\rangle
\end{aligned}$$

$$\leq \left(1-2(\tilde{\gamma}-\alpha\gamma)\alpha_n\right)\|w_n-\tilde{w}\|^2+\alpha_n\Big\{2\left\langle Tw_n-\tilde{w},\gamma f(\tilde{w})-A\tilde{w}\right\rangle$$
$$+\alpha_n\|\gamma f(w_n)-A\tilde{w}\|^2-2\alpha_n\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|\gamma f(w_n)-A\tilde{w}\|$$
$$+\alpha_n\tilde{\gamma}^2\|w_n-\tilde{w}\|^2\Big\}+\beta_n\Big[\Big\{2\left\langle Tw_n-\tilde{w},u_n-A\tilde{w}\right\rangle+\beta_n\|u_n-A\tilde{w}\|^2$$
$$-2\beta_n\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|u_n-A\tilde{w}\|+\beta_n\tilde{\gamma}^2\|w_n-\tilde{w}\|^2\Big\}$$
$$+2\Big\{\|\gamma f(w_n)-A\tilde{w}\|\cdot\|u_n-A\tilde{w}\|-\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|\gamma f(w_n)-A\tilde{w}\|$$
$$-\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|u_n-A\tilde{w}\|+2\gamma^2\|w_n-\tilde{w}\|^2\Big\}\Big].$$

Since $\{w_n\},\{f(w_n)\},\{ATw_n\},\{u_n\}$ are bounded, $\alpha_n\to 0$ and $\sum_{n=1}^{\infty}\beta_n<\infty.$
We obtain

$$\|w_{n+1}-\tilde{w}\|^2 = (1-\lambda_n)\|w_n-\tilde{w}\|^2+b_n+c_n, \tag{26}$$

where

$$\lambda_n = 2(\tilde{\gamma}-\alpha\gamma)\alpha_n,$$
$$b_n = \alpha_n\Big\{2\left\langle Tw_n-\tilde{w},\gamma f(\tilde{w})-A\tilde{w}\right\rangle+\alpha_n\|\gamma f(w_n)-A\tilde{w}\|^2$$
$$-2\alpha_n\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|\gamma f(w_n)-A\tilde{w}\|+\alpha_n\tilde{\gamma}^2\|w_n-\tilde{w}\|^2\Big\},$$
$$c_n = \beta_n\Big[\Big\{2\left\langle Tw_n-\tilde{w},u_n-A\tilde{w}\right\rangle+\beta_n\|u_n-A\tilde{w}\|^2-2\beta_n\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|u_n-A\tilde{w}\|$$
$$+\beta_n\tilde{\gamma}^2\|w_n-\tilde{w}\|^2\Big\}+2\Big\{\|\gamma f(w_n)-A\tilde{w}\|\cdot\|u_n-A\tilde{w}\|$$
$$-\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|\gamma f(w_n)-A\tilde{w}\|-\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|u_n-A\tilde{w}\|+2\gamma^2\|w_n-\tilde{w}\|^2\Big\}\Big].$$

We note by $(C3)$ and (25) that $\sum_{n=0}^{\infty}\lambda_n<\infty,$

$$b_n/\lambda_n = 2\left\langle Tw_n-\tilde{w},\gamma f(\tilde{w})-A\tilde{w}\right\rangle+\alpha_n\|\gamma f(w_n)-A\tilde{w}\|^2$$
$$-2\alpha_n\left\|A\left(Tw_n-\tilde{w}\right)\right\|\cdot\|\gamma f(w_n)-A\tilde{w}\|+\alpha_n\tilde{\gamma}^2\|w_n-\tilde{w}\|^2\big/2(\tilde{\gamma}-\alpha\gamma)\to 0,$$

and $\sum_{n=0}^{\infty}c_n<\infty.$
Now applying Lemma 2 to (26), we can conclude that $w_n\to\tilde{w}.$

■

If $\{\beta_n\}\equiv 0$ and $T:H\to H$ is a nonexpansive mapping in Theorem 10, we get,

Corollary 11. (Marino and Xu, 2006). Let $\{x_n\}$ be generated by algorithm:

$$x_{n+1} = (I-\alpha_n A)Tx_n+\alpha_n\gamma f(x_n), \quad n\geq 0.$$

Assume the sequence $\{\alpha_n\}$ of parameters satisfying conditions $(C1)-(C3)$.
Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Theorem 8.

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